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Riverside Mathematical Monographs

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THE REORGANIZATION OF MATHEMATICS IN SECONDARY EDUCATION (PART I)

A REPORT BY THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS UNDER THE AUSPICES OF THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

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EDITOR'S INTRODUCTION

THE report of the National Committee on Mathematical Requirements, entitled the *Reorganization of Mathematics in Secondary Education*, was published as a volume of over 600 pages in 1923. It is generally recognized that this report has had and continues to have a profound influence on the teaching of secondary school mathematics in this country. The large edition of 25,000 copies of this report was exhausted in the early spring of 1926. Demands for the report have continued since that time. It is therefore a source of gratification that Part I of this report, containing all the recommendations of the Committee, is now issued as one of the Riverside Mathematical Monographs.

The text of the report is a *verbatim* reprint of the original text (except for the correction of obvious misprints). The National Committee was discharged in December, 1923, and there is therefore no one in existence who has authority to revise the report. It has seemed desirable, however, to add some comments in the form of footnotes. These additions have all been enclosed in square brackets. Some supplementary material has been added in an appendix. This consists in part of significant extracts from Part II of the report, and in part of brief statements prepared especially for this edition.

I should like to express my gratitude to all those who have helped in the publication of the present Monograph. My special thanks are due to the former members of the

National Committee, and particularly to Professor David Eugene Smith and Miss Vevia Blair; to Professor Thomas S. Fiske, Secretary of the College Entrance Examination Board, for his permission to reprint the Board's definitions of college entrance requirements; and to Professor C. B. Upton for his statement on standardized tests which will be found in the Appendix.

J. W. Young

THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS

(Under the auspices of
The Mathematical Association of America, Inc.)

OFFICERS

J. W. Young, chairman, Dartmouth College, Hanover, New Hampshire.

J. A. Foberg, vice chairman, State Department of Public Instruction, Harrisburg, Pennsylvania.

MEMBERS

A. R. Crathorne, University of Illinois.

C. N. Moore, University of Cincinnati.[1]

E. H. Moore, University of Chicago.

David Eugene Smith, Columbia University.

H. W. Tyler, Massachusetts Institute of Technology.

J. W. Young, Dartmouth College.

W. F. Downey, English High School, Boston, Massachusetts, representing the Association of Teachers of Mathematics in New England.[2]

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[1] Professor Moore took the place vacated in 1918 by the resignation of Oswald Veblen, Princeton University.

[2] Mr. Downey took the place vacated in 1919 by the resignation of G. W. Evans, Charlestown High School, Boston, Massachusetts.

THE NATIONAL COMMITTEE

Vevia Blair, Horace Mann School, New York City, representing the Association of Teachers of Mathematics in the Middle States and Maryland.

J. A. Foberg, director of mathematical instruction, State Department, Harrisburg, Pennsylvania,[3] representing the Central Association of Science and Mathematics Teachers.

A. C. Olney, Commissioner of Secondary Education, Sacramento, California.

Raleigh Schorling, The Lincoln School, New York City.

P. H. Underwood, Ball High School, Galveston, Texas.

Eula A. Weeks, Cleveland High School, St. Louis, Missouri.

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[3] Until July, 1921, of the Crane Technical High School, Chicago, Illinois.

PREFACE TO THE FIRST EDITION

THE National Committee on Mathematical Requirements was organized in the late summer of 1916 under the auspices of The Mathematical Association of America for the purpose of giving national expression to the movement for reform in the teaching of mathematics, which had gained considerable headway in various parts of the country, but which lacked the power that coördination and united effort alone could give.

The original nucleus of the committee, appointed by E. R. Hedrick, then president of the Association, consisted of the following: A. R. Crathorne, University of Illinois; E. H. Moore, University of Chicago; D. E. Smith, Columbia University; H. W. Tyler, Massachusetts Institute of Technology; Oswald Veblen, Princeton University; and J. W. Young, Dartmouth College, chairman. This committee was instructed to add to its membership so as to secure adequate representation of secondary school interests, and then to undertake a comprehensive study of the whole problem concerned with the improvement of mathematical education and to cover the field of secondary and collegiate mathematics.

This group held its first meeting in September, 1916, at Cambridge, Massachusetts. At that meeting it was decided to ask each of the three large associations of secondary school teachers of mathematics (The Association of Teachers of Mathematics in New England, The Association of Teachers of Mathematics in the Middle States

and Maryland, and The Central Association of Science and Mathematics Teachers) to appoint an official representative on the committee. At this time also a general plan for the work of the committee was outlined and agreed upon.

In response to the request above referred to the following were appointed by the respective associations: Miss Vevia Blair, Horace Mann School, New York City, representing the Middle States and Maryland Association; G. W. Evans, Charlestown High School, Boston, Massachusetts, representing the New England Association;[1] and J. A. Foberg, Crane Technical High School, Chicago, Illinois, representing the Central Association.

At later dates the following members were appointed: A. C. Olney, Commissioner of Secondary Education, Sacramento, California; Raleigh Schorling, The Lincoln School, New York City; P. H. Underwood, Ball High School, Galveston, Texas; and Miss Eula A. Weeks, Cleveland High School, Saint Louis, Missouri.

From the very beginning of its deliberations the committee felt that the work assigned to it could not be done effectively without adequate financial support. The wide geographical distribution of its membership made a full attendance at meetings of the committee difficult if not impossible without financial resources sufficient to defray the traveling expenses of members, the expenses of clerical assistance, etc. Above all, it was felt that, in order to give to the ultimate recommendations of the committee the authority and effectiveness which they should have,

[1] Mr. Evans resigned in the summer of 1919, owing to an extended trip abroad; his place was taken by W. F. Downey, English High School, Boston, Massachusetts.

it was necessary to arouse the interest and secure the active coöperation of teachers, administrators, and organizations throughout the country — that the work of the committee should represent a coöperative effort on a truly national scale.

For over two years, owing in large part to the World War, attempts to secure adequate financial support proved unsuccessful. Inevitably also the war interfered with the committee's work. Several members were engaged in war work [2] and the others were carrying extra burdens on account of such work undertaken by their colleagues.

In the spring of 1919, however, and again in 1920, the committee was fortunate in securing generous appropriations from the General Education Board of New York City for the prosecution of its work.[3]

This made it possible greatly to extend the committee's activities. The work was planned on a large scale for the purpose of organizing a truly nation-wide discussion of the problems facing the committee, and J. W. Young and J. A. Foberg were selected to devote their whole time to the work of the committee. Suitable office space was secured and adequate stenographic and clerical help was employed.

The results of the committee's work and deliberations are presented in the following report. A word as to the

[2] Professor Veblen resigned in 1917 on account of the pressure of his war duties. His place was taken on the committee by Professor C. N. Moore, University of Cincinnati.

[3] Again in November, 1921, the General Education Board made appropriations to cover the expense of publishing and distributing the present report and to enable the committee to carry on certain phases of its work during the years 1922–23.

methods employed may, however, be of interest at this point. The committee attempted to establish working contact with all organizations of teachers and others interested in its problems and to secure their active assistance. Nearly 100 such organizations have taken part in this work. A list of these organizations will be found in the appendix to this report (p. 632).[4] Provisional reports on various phases of the problem were submitted to these coöperating organizations in advance of publication, and criticisms, comments, and suggestions for improvement were invited from individuals and special coöperating committees. The reports previously published for the committee by the United States Bureau of Education [5] and in *The Mathematics Teacher* [6] and designated as "preliminary" are the result of this kind of coöperation. The value of such assistance can hardly be overestimated and the committee desires to express to all individuals, organizations, and educational journals that have taken part its hearty appreciation and thanks. The committee believes it is safe to say, in view of the methods used in formulating them, that the recommendations of this final report have the approval of the great majority of progressive teachers throughout the country.

[4] [Not contained in the present edition.]

[5] *The Reorganization of the First Courses in Secondary School Mathematics*, U.S. Bureau of Education, Secondary School Circular, no. 5, February, 1920. 11 pp. *Junior High School Mathematics*, U.S. Bureau of Education, Secondary School Circular, no. 6, July, 1920. 10 pp. *The Function Concept in Secondary School Mathematics*, Secondary School Circular, no. 8, June, 1921. 10 pp.

[6] "Terms and Symbols in Elementary Mathematics," *The Mathematics Teacher*, vol. 14 (March, 1921), pp. 107–18. "Elective Courses in Mathematics for Secondary Schools," *The Mathematics Teacher*, vol. 14 (April, 1921), pp. 161–70. "College Entrance Requirements in Mathematics," *The Mathematics Teacher*, vol. 14 (May, 1921), pp. 224–45.

PREFACE TO THE FIRST EDITION

No attempt has been made in this report to trace the origin and history of the various proposals and movements for reform nor to give credit either to individuals or organizations for initiating them. A convenient starting point for the history of the modern movement in this country may be found in E. H. Moore's presidential address before the American Mathematical Society in 1902. But the movement here is only one manifestation of a movement that is world-wide and in which very many individuals and organizations have played a prominent part. The student interested in this phase of the subject is referred to the extensive publications of the International Commission on the Teaching of Mathematics, to the *Bibliography of the Teaching of Mathematics*, 1900–12, by D. E. Smith and C. Goldziher (U.S. Bureau of Education, Bulletin, 1912, no. 29) and to the bibliography (since 1912) to be found in this report (chap. XVI, p. 539*f*).[8]

The National Committee expects to maintain its office, with a certain amount of clerical help, during the year 1922–23 and perhaps for a longer period. It is hoped that in this way it may continue to serve as a clearing house for all activities looking to the improvement of the teaching of mathematics in this country, and to assist in bringing about the effective adoption in practice of the recommendations made in the following report, with such modifications of them as continued study and experimentation may show to be desirable.

[7] E. H. Moore: *On the Foundations of Mathematics*, Bulletin of the American Mathematical Society, vol. 9 (1902–03), p. 402; *Science*, vol. 17, p. 401.

[8] [Not contained in the present edition.]

CONTENTS

THE REORGANIZATION OF MATHEMATICS IN SECONDARY EDUCATION

I

A BRIEF OUTLINE OF THE REPORT

The present chapter gives a brief general outline of the contents of this report for the purpose of orienting the reader and making it possible for him to gain quickly an understanding of its scope and the problems which it considers.

The valid aims and purposes of instruction in mathematics are considered in Chapter II. A formulation of such aims and a statement of general principles governing the committee's work is necessary as a basis for the later specific recommendations. Here will be found the reasons for including mathematics in the course of study for all secondary school pupils.

To the end that all pupils in the period of secondary education shall gain early a broad view of the whole field of elementary mathematics, and, in particular, in order to insure contact with this important element in secondary education on the part of the very large number of pupils who, for one reason or another, drop out of school by the end of the ninth year, the National Committee recommends emphatically that the course of study in mathematics during the seventh, eighth, and ninth years contain the fundamental notions of arithmetic, of algebra, of intuitive geometry, of numerical trigonometry, and at

least an introduction to demonstrative geometry, and that this body of material be required of all secondary school pupils.

A detailed account of this material is given in Chapter III. Careful study of the later years of our elementary schools, and comparison with European schools, have shown the vital need of reorganization of mathematical instruction, especially in the seventh and eighth years. The very strong tendency now evident to consider elementary education as ceasing at the end of the sixth school year, and to consider the years from the seventh to the twelfth inclusive as comprising years of secondary education, gives impetus to the movement for reform of the teaching of mathematics at this stage.

While Chapter III is devoted to a consideration of the body of materials of instruction in mathematics that is regarded as of sufficient importance to form part of the course of study for all secondary school pupils, Chapter IV is devoted to a consideration of the types of material that properly enter into courses of study for pupils who continue their study of mathematics beyond the minimum regarded as essential for all pupils. Here will be found recommendations concerning the traditional subject-matter of the tenth, eleventh, and twelfth school years, and also certain material that heretofore has been looked upon in this country as belonging rather to college courses of study; as, for instance, the elementary ideas and processes of the calculus.

Chapter V is devoted to a study of the types of secondary school instruction in mathematics that may be looked upon as furnishing the best preparation for successful work in college. This study leads to the conclu-

sion that there is no conflict between the needs of those pupils who ultimately go to college and those who do not. Certain very definite recommendations are made as to changes that appear desirable in the statement of college entrance requirements and in the type of college entrance examination.[1]

Chapter VI contains lists of propositions and constructions in plane and in solid geometry. The propositions are classified in such a way as to separate from others of less importance those which are regarded as so fundamental that they should form the common minimum of any standard course in the subject.[2]

The statement previously made in preliminary reports of the National Committee and repeated in Chapter II, that the function concept should serve as a unifying element running throughout the instruction in the mathematics of the secondary school, has brought many requests for a more precise definition of the rôle of the function concept in secondary school mathematics. Chapter VII is intended to meet this demand.

[[1] Such changes have, since the publication of this report, been incorporated in the requirements formulated by the College Entrance Examination Board. In order to facilitate comparison with the recommendations of the National Committee the definitions of the C.E.E.B. now in force will be found in this monograph. (See Chapter VI and Appendix.) They have been taken with permission from Document 107 (Definition of the Requirements in Elementary Algebra, Advanced Algebra, Trigonometry) and Document 108 (Definition of the Requirements in Plane and Solid Geometry), published by the College Entrance Examination Board, 431 West 117th Street, New York City. Copies of these pamphlets, which contain much valuable information, will be mailed by the Board free of charge to any teacher.]

[[2] The College Entrance Examination Board has also published such lists (see Document 108 referred to in the previous note). The lists in Chapter VI (pp. 79–91) indicate the differences between the National Committee and the C.E.E.B. lists.]

OUTLINE OF THE REPORT

Recommendations as to the adoption and use of terms and symbols in elementary mathematics are contained in Chapter VIII. It is intended to present a norm embodying agreement as to best current practice.

The remaining chapters[3] give for the most part the results of special investigations conducted for the National Committee. The contents of these chapters are indicated sufficiently in the general table of contents and in the tables of contents preceding many of the chapters in question.

[[3] With the exception of certain extracts (see Appendix), these chapters are not included in the present edition.]

II

AIMS OF MATHEMATICAL INSTRUCTION — GENERAL PRINCIPLES

I. Introduction

A discussion of mathematical education, and of ways and means of enhancing its value, must be approached first of all on the basis of a precise and comprehensive formulation of the valid aims and purposes of such education.[1] Only on such a basis can we approach intelligently the problems relating to the selection and organization of material, the methods of teaching and the point of view which should govern the instruction, and the qualifications and training of the teachers who impart it. Such aims and purposes of the teaching of mathematics, moreover, must be sought in the nature of the subject, the rôle it plays in the practical, intellectual, and spiritual life of the world, and in the interests and capacities of the students.

Before proceeding with the formulation of these aims, however, we may properly limit to some extent the field of our enquiry. We are concerned primarily with the period of secondary education — comprising, in the modern

[1] Reference may here be made to the formulation of the principal aims in education to be found in the *Cardinal Principles of Secondary Education*, published by the U.S. Bureau of Education as Bulletin no. 55, 1918. The main objectives of education are there stated to be: (1) health; (2) command of fundamental processes; (3) worthy home membership; (4) vocation; (5) citizenship; (6) worthy use of leisure; (7) ethical character. These objectives are held to apply to all education — elementary, secondary, and higher — and all subjects of instruction are to contribute to their achievement.

junior and senior high schools, the period beginning with the seventh and ending with the twelfth school year, and concerning itself with pupils ranging in age normally from 12 to 18 years. References to the mathematics of the grades below the seventh (mainly arithmetic) and beyond the senior high school will be only incidental.

Furthermore, we are primarily concerned at this point with what may be described as "general" aims, that is to say, aims which are valid for large sections of the school population and which may properly be thought of as contributing to a general education as distinguished from the specific needs of vocational, technical, or professional education.

II. The Aims of Mathematical Instruction

With these limitations in mind we may now approach the problem of formulating the more important aims that the teaching of mathematics should serve. It has been customary to distinguish three classes of aims: (1) Practical or utilitarian, (2) disciplinary, (3) cultural; and such a classification is indeed a convenient one. It should be kept clearly in mind, however, that the three classes mentioned are not mutually exclusive and that convenience of discussion rather than logical necessity often assigns a given aim to one or the other of these classes. Indeed, any truly disciplinary aim is practical and, in a broad sense, the same is true of cultural aims.

Practical aims. By a practical or utilitarian aim, in the narrower sense, we mean then the immediate or direct usefulness in life of a fact, method, or process in mathematics.

1. The immediate and undisputed utility of the *funda-*

mental processes of arithmetic in the life of every individual demands our first attention. The first instruction in these processes, it is true, falls outside the period of instruction which we are considering. By the end of the sixth grade the child should be able to carry out the four fundamental operations with integers and with common and decimal fractions accurately and with a fair degree of speed. This goal can be reached in all schools — as it is being reached in many — if the work is done under properly qualified teachers and if drill is confined to the simpler cases which alone are of importance in the practical life of the great majority. (See more specifically, Chapter III, p. 29.) Accuracy and facility in numerical computation are of such vital importance, however, to every individual that effective drill in this subject should be continued throughout the secondary school period, not in general as a separate topic, but in connection with the numerical problems arising in other work. In this numerical work, besides accuracy and speed, the following aims are of the greatest importance:

(*a*) A progressive increase in the pupil's understanding of the nature of the fundamental operations and power to apply them in new situations. The fundamental laws of algebra are a potent influence in this direction. (See 3, below.)

(*b*) Exercise of common sense and judgment in computing from approximate data, familiarity with the effect of small errors in measurements, the determination of the number of figures to be used in computing and to be retained in the result, and the like.[2]

[[2] The extent to which this recommendation has found its way into recent textbooks and the emphasis these topics have received constitute

(*c*) The development of self-reliance in the handling of numerical problems through the consistent use of checks on all numerical work.

2. Of almost equal importance to every educated person is *an understanding of the language of algebra* and the ability to use this language intelligently and readily in the expression of such simple quantitative relations as occur in every-day life and in the normal reading of the educated person.

Appreciation of the significance of formulas [3] and ability to work out simple problems by setting up and solving the necessary equations must nowadays be included among the minimum requirements of any program of universal education.

3. The development of the ability to understand and to use such elementary algebraic methods involves a study of the *fundamental laws of algebra* and at least a certain minimum of drill in algebraic technique, which, when properly taught, will furnish the foundation for an understanding of the significance of the processes of arithmetic already referred to. The essence of algebra as distinguished from arithmetic lies in the fact that algebra concerns itself with the operations upon numbers *in general*, while arithmetic confines itself to operations on *particular* numbers.

4. The ability to understand and interpret correctly *graphic representations* of various kinds, such as nowadays abound in popular discussions of current scientific, social,

one of the notable changes in the teaching of mathematics of the last few years.]

[[3] The importance of the formula has received recognition by explicit inclusion in the C.E.E.B. requirements in Elementary Algebra.] See p. 140.

industrial, and political problems, will also be recognized as one of the necessary aims in the education of every individual. This applies to the representation of statistical data which are becoming increasingly important in the consideration of our daily problems, as well as to the representation and understanding of various sorts of dependence of one variable quantity upon another.

5. Finally, among the practical aims to be served by the study of mathematics should be listed familiarity with the *geometric forms* common in nature, industry, and life; the elementary properties and relations of these forms, including their *mensuration;* the development of *space-perception;* and the exercise of *spatial imagination.* This involves acquaintance with such fundamental ideas as congruence and similarity and with such fundamental facts as those concerning the sum of the angles of a triangle, the pythagorean proposition, and the areas and volumes of the common geometric forms.

Among directly practical aims should also be included the acquisition of the ideas and concepts in terms of which the quantitative thinking of the world is done, and of ability to think clearly in terms of those concepts. It seems more convenient, however, to discuss this aim in connection with the disciplinary aims.

Disciplinary aims. We would include here those aims which relate to mental training, as distinguished from the acquisition of certain specific skills discussed in the preceding section. Such training involves the development of certain more or less general characteristics and the formation of certain mental habits which, besides being directly applicable in the setting in which they are developed or formed, are expected to operate also in more

or less closely related fields — that is, to "transfer" to other situations.

The subject of the transfer of training has for a number of years been a very controversial one. Only recently has there been any evidence of agreement among the body of educational psychologists. We need not at this point go into detail as to the present status of disciplinary values since this forms the subject of a separate chapter (Chapter IX; see also Chapter X).[4] It is sufficient for our present purpose to call attention to the fact that most psychologists have abandoned two extreme positions as to transfer of training. The first asserted that a pupil trained to reason well in geometry would thereby be trained to reason equally well in any other subject; the second denied the possibility of any transfer and hence the possibility of any general mental training. That the effects of training do transfer from one field of learning to another is now, however, recognized. The amount of

[[4] Not included in the present edition. See, however, the extract from Chapter IX on p. 123.

In connection with the latter part of this paragraph, the following opinion expressed by Professor W. C. Bagley, and quoted from pp. 97 and 98 of the complete report is significant:

The evidence from the experiments is clearly in favor of a certain amount of transfer.

In my opinion, the possibilities of transfer are increased by the kind of teaching that makes the student conscious of the procedure as such, and keenly appreciative of its value as a *general* procedure.

The theory of "identical elements" I regard as sound — but it is not rich in pedagogical suggestiveness. The theory of transfer through "concepts of method," and "ideals of procedure" furnishes a definite suggestion for teaching. The two theories are not inconsistent with one another: a person who has gained an understanding and an *appreciation* of a procedure will not be limited to the "identical elements" which come by accident; he will *search* for identities — for places at which the procedure may be applied.]

transfer in any given case depends upon a number of conditions. If these conditions are favorable, there may be considerable transfer, but in any case the amount of transfer is difficult to measure. Training in connection with certain attitudes, ideals, and ideas is now almost universally admitted by psychologists to have general value. It may, therefore, be said that, with proper restrictions, general mental discipline is a valid aim in education.

The aims which we are discussing are so important in the restricted domain of quantitative and spatial (i.e., mathematical or partly mathematical) thinking which every educated individual is called upon to perform that we do not need for the sake of our argument to raise the question as to the extent of transfer to less mathematical situations.

In formulating the disciplinary aims of the study of mathematics the following should be mentioned:

(1) The acquisition, in precise form, of those *ideas or concepts in terms of which the quantitative thinking of the world is done.* Among these ideas and concepts may be mentioned ratio and measurement (lengths, areas, volumes, weights, velocities, and rates in general, etc.), proportionality and similarity, positive and negative numbers, and the dependence of one quantity upon another.

(2) The development of *ability to think clearly in terms of such ideas and concepts.* This ability involves training in

(*a*) Analysis of a complex situation into simpler parts. This includes the recognition of essential factors and the rejection of the irrelevant.

(*b*) The recognition of logical relations between inter-

dependent factors and the understanding and, if possible, the expression of such relations in precise form.

(*c*) Generalization; that is, the discovery and formulation of a general law and an understanding of its properties and applications.

(3) The acquisition of *mental habits and attitudes* which will make the above training effective in the life of the individual. Among such habitual reactions are the following: a seeking for relations and their precise expression; an attitude of inquiry; a desire to understand, to get to the bottom of a situation; concentration and persistence; a love for precision, accuracy, thoroughness, and clearness, and a distaste for vagueness and incompleteness; a desire for orderly and logical organization as an aid to understanding and memory.

(4) Many of these disciplinary aims are included in the broad sense of *the idea of relationship or dependence* — in what the mathematician in his technical vocabulary refers to as a "function" of one or more variables. Training in "functional thinking," that is, thinking in terms of and about relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics.

Cultural aims. By cultural aims we mean those somewhat less tangible but none the less real and important intellectual, ethical, esthetic or spiritual aims that are involved in the development of appreciation and insight and the formation of ideals of perfection. As will be at once apparent the realization of some of these aims must await the later stages of instruction, but some of them may and should operate at the very beginning.

More specifically we may mention the development or acquisition of

(1) *Appreciation of beauty* in the geometrical forms of nature, art, and industry.

(2) *Ideals of perfection* as to logical structure, precision of statement and of thought, logical reasoning (as exemplified in the geometric demonstration), discrimination between the true and the false, etc.

(3) *Appreciation of the power of mathematics* — of what Byron expressively called "the power of thought, the magic of the mind" [5] — and the rôle that mathematics and abstract thinking, in general, have played in the development of civilization; in particular in science, in industry, and in philosophy. In this connection mention should be made of the religious effect, in the broad sense, which the study of the infinite and of the permanence of laws in mathematics tends to establish.[6]

III. The Point of View Governing Instruction

The practical aims enumerated above, in spite of their vital importance, may without danger be given a secondary position in seeking to formulate the general point of view which should govern the teacher, provided only that they receive due recognition in the selection of material and that the necessary minimum of technical drill is insisted upon.

The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of

[5] D. E. Smith: "Mathematics in the Training for Citizenship," *Teachers College Record*, vol. 18 (May, 1917), p. 6.

[6] For an elaboration of the ideas here presented in the barest outline, the reader is referred to the article by D. E. Smith already mentioned and to his presidential address before the Mathematical Association of America, "Religio Matematici," *American Mathematical Monthly*, vol. 28 (October, 1921), pp. 339–49, also published in *The Mathematics Teacher*, vol. 14 (December, 1921), pp. 413–26.

analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

All topics, processes, and drill in technique which do not directly contribute to the development of the powers mentioned should be eliminated from the curriculum. It is recognized that in the earlier periods of instruction the strictly logical organization of subject-matter [7] is of less importance than the acquisition, on the part of the pupil, of experience as to facts and methods of attack on significant problems, of the power to see relations, and of training in accurate thinking in terms of such relations. Care must be taken, however, through the dominance of the course by certain general ideas that it does not become a collection of isolated and unrelated details.

Continued emphasis throughout the course must be placed on the development of ability to grasp and to utilize ideas, processes, and principles in the solution of concrete problems rather than on the acquisition of mere facility or skill in manipulation. The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. On the side of algebra, the ability to understand its language and to use it intelligently, the ability to analyze a problem, to formulate it mathematically, and to interpret the result must be dominant aims. *Drill in algebraic manipulation should*

[7] Dewey, *How We Think*, p. 62. "The logical from the standpoint of subject-matter represents the goal, the last term of training, not the point of departure."

be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take. It must be conceived throughout as a means to an end, not as an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable students to carry out the essential processes accurately and expeditiously.[8]

On the side of geometry the formal demonstrative work should be preceded by a reasonable amount of informal work of an intuitive, experimental, and constructive character. Such work is of great value in itself; it is needed also to provide the necessary familiarity with geometric ideas, forms, and relations, on the basis of which alone intelligent appreciation of formal demonstrative work is possible.

The one great idea which is best adapted to unify the course is that of the *functional relation*. The concept of a variable and of the dependence of one variable upon another is of fundamental importance to every one. It is true that the general and abstract forms of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and

[[8] The experience of the last few years has fully confirmed this recommendation regarding the desirability of eliminating the former *excessive* emphasis on complicated algebraic technique. The authors of some recent texts have perhaps gone to the opposite extreme, by not taking sufficiently to heart the caution contained in the last section of the above paragraph.]

illustrations of dependence even in the early parts of the course. Means to this end will be found in connection with the tabulation of data and the study of the formula and of the graph and of their uses.

The primary and underlying principle of the course should be the idea of relationship between variables, including the methods of determining and expressing such relationship. The teacher should have this idea constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the formation of the general concept of functionality depends. (For a more detailed discussion of these ideas see Chapter VII.)

The general ideas which appear more explicitly in the course, and under the dominance of one or another of which all topics should be brought, are: (1) The formula, (2) graphic representation, (3) the equation, (4) measurement and computation, (5) congruence and similarity, (6) demonstration. These are considered in more detail in a later section of the report (Chapters III and IV).

IV. The Organization of Subject-Matter

"General" courses. We have already called attention to the fact that, in the earlier periods of instruction especially, logical principles of organization are of less importance than psychological and pedagogical principles. In recent years there has developed among many progressive teachers a very significant movement away from the older rigid division into "subjects" such as arithmetic, algebra, and geometry, each of which shall be

"completed" before another is begun, and toward a rational breaking down of the barriers separating these subjects, in the interest of an organization of subject-matter that will offer a psychologically and pedagogically more effective approach to the study of mathematics.

There has thus developed the movement toward what are variously called "composite," "correlated," "unified," or "general" courses. The advocates of this new method of organization base their claims on the obvious and important interrelations between arithmetic, algebra, and geometry (mainly intuitive), which the student must grasp before he can gain any real insight into mathematical methods and which are inevitably obscured by a strict adherence to the conception of separate "subjects." The movement has gained considerable new impetus by the growth of the junior high school, and there can be little question that the results already achieved by those who are experimenting with the new methods of organization warrant the abandonment of the extreme "water-tight compartment" method of presentation.

The newer method of organization enables the pupil to gain a broad view of the whole field of elementary mathematics early in his high-school course. In view of the very large number of pupils who drop out of school at the end of the eighth or the ninth school year or who for other reasons then cease their study of mathematics, this fact offers a weighty advantage over the older type of organization under which the pupils studied algebra alone during the ninth school year, to the complete exclusion of all contact with geometry.

It should be noted, however, that the specific recom-

mendations as to content given in the next two chapters do not necessarily imply the adoption of a different type of organization of the materials of instruction. A large number of high schools will for some time continue to find it desirable to organize their courses of study in mathematics by subjects — algebra, plane geometry, etc. Such schools are urged to adopt the recommendations made with reference to the content of the separate subjects. These, in the main, constitute an essential simplification as compared with present practice. The economy of time that will result in courses in ninth-year algebra, for instance, will permit of the introduction of the newer type of material, including intuitive geometry and numerical trigonometry, and thus the way will be prepared for the gradual adoption in larger measure of the recommendations of this report.

At the present time it is not possible to designate any particular order of topics or any organization of the materials of instruction as being the best or as calculated most effectively to realize the aims and purposes here set forth. More extensive and careful experimental work must be done by teachers and administrators before any such designation can be made that shall avoid undesirable extremes and that shall bear the stamp of general approval. This experimental work will prove successful in proportion to the skill and insight exercised in adapting the aims and purposes of instruction to the interests and capacities of the pupils. One of the greatest weaknesses of the traditional courses is the fact that both the interests and the capacities of pupils have received insufficient consideration and study. For a detailed account of courses in mathematics at a number of the most successful experi-

mental schools the reader is referred to Chapter XII[9] of this report.

Required courses. The National Committee believes that the material described in the next chapter should be required of all pupils and that under favorable conditions this minimum of work can be completed by the end of the ninth school year. In the junior high school, comprising grades seven, eight, and nine, the course for these three years should be planned as a unit *with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little reference to courses which he may or may not take in succeeding years.* In particular, college entrance requirements should, during these years, receive no specific consideration. Fortunately there appears to be no conflict of interest during this period between those pupils who ultimately go to college and those who do not; a course planned in accordance with the principle just enunciated will form a desirable foundation for college preparation.[10] (See Chapter V; also, the experience of the schools described in Chapter XII[9]).

Similarly, in case of the at present more prevalent 8–4 school organization, the mathematical material of the seventh and eighth grades should be selected and organized as a unit with the same purpose; the same applies to the work of the first year (ninth grade) of the standard four-year high school, and to later years in which mathematics may be a required subject.

[[9] Not included in the present edition.]

[[10] It may be added that the present requirements of the College Entrance Examination Board are entirely in harmony with the spirit of the present report, so that the suggestions above made are now in no way in conflict with the demands of the colleges.]

In the case of some elective courses the principle needs to be modified so as to meet whatever specific vocational or technical purposes the courses may have. (See Chapter IV.)

The movement toward correlation of the work in mathematics with other courses in the curriculum, notably those in science, is as yet in its infancy. The results of such efforts will be watched with the keenest interest.

The junior high-school movement. Reference has several times been made to the junior high school. The National Committee adopted the following resolution on April 24, 1920:

The National Committee approves the junior high school form of organization, and urges its general adoption in the conviction that it will secure greater efficiency in the teaching of mathematics.

The committee on the reorganization of secondary education, appointed by the National Education Association, in its pamphlet on the *Cardinal Principles of Secondary School Education*, issued in 1918 by the Bureau of Education, advocates an organization of the school system whereby the first six years shall be devoted to elementary education and the following six years to secondary education to be divided into two periods which may be designated as junior and senior periods.

To those interested in the study of the questions relating to the history and present status of the junior high school movement, the following books are recommended: *Principles of Secondary Education*, by Inglis (Houghton Mifflin Company, 1918); "The Junior High School," *The Fifteenth Yearbook* (Part III) of the National Society for the Study of Education (Public School Publishing Com-

pany, 1919); *The Junior High School*, by Bennett (Warwick & York, 1919); *The Junior High School*, by Briggs (Houghton Mifflin Company, 1920); and *The Junior High School*, by Koos (Harcourt, Brace & Co., 1920).[11]

V. The Training of Teachers

While the greater part of this report concerns itself with the content of courses in mathematics, their organization and the point of view which should govern the instruction, and investigations relating thereto, the National Committee must emphasize strongly its conviction that even more fundamental is the problem of the teacher — his qualifications and training, his personality, skill, and enthusiasm.

The greater part of the failure of mathematics is due to poor teaching. Good teachers have in the past succeeded, and will continue to succeed, in achieving highly satisfactory results with the traditional material; poor teachers will not succeed even with the newer and better material.

The United States is far behind Europe in the scientific and professional training required of its secondary school teachers (see Chapter XIV[12]). The equivalent of two or three years of graduate and professional training in addition to a general college course is the normal requirement

[[11] Other books on the junior high school movement, published since the publication of this report, are: *Junior High School Procedure*, by Touton and Struthers (Ginn & Co.); the monograph on *Junior High School Practices*, by Glass (University of Chicago Press); *Junior High School Curricula*, by Hines (The Macmillan Company); *Junior High School Mathematics*, by Barber (Houghton Mifflin Company); *Psychology of the Junior High School*, by Pechstein and McGregor (Houghton Mifflin Company); *The Junior High School*, by Smith (The Macmillan Company).]

[[12] Not included in the present edition. But see extract on p. 126.]

for secondary school teachers in most European countries. Moreover, the recognized position of the teacher in the community must be such as to attract men and women of the highest ability into the profession. This means not only higher salaries but smaller classes and more leisure for continued study and professional advancement. It will doubtless require a considerable time before the public can be educated to realize the wisdom of taxing itself sufficiently to bring about the desired result. But if this ideal is continually advanced and supported by sound argument there is every reason to hope that in time the goal may be reached.

In the meantime everything possible should be done to improve the present situation. One of the most vicious and widespread practices consists in assigning a class in mathematics to a teacher who has had no special training in the subject and whose interests lie elsewhere, because in the construction of the time schedule he or she happens to have a vacant period at the time. This is done on the principle, apparently, that "anybody can teach mathematics" by simply following a textbook and devoting ninety per cent of the time to drill in algebraic manipulation or to the recitation of the memorized demonstration of theorems in geometry.

It will be apparent from the study of this report that a successful teacher of mathematics must not only be highly trained in his subject and have a genuine enthusiasm for it but must have also peculiar attributes of personality and above all insight of a high order into the psychology of the learning process as related to the higher mental activities. Administrators should never lose sight of the fact that while mathematics if properly taught is one of

the most important, interesting, and valuable subjects of the curriculum, it is also one of the most difficult to teach successfully.

Standards for teachers. It is necessary at the outset to make a fundamental distinction between standards in the sense of requirements for appointment to teaching positions, and standards of scientific attainment which shall determine the curricula of colleges and normal schools aiming to give candidates the best practicable preparation. The former requirements should be high enough to insure competent teaching, but they must not be so high as to form a serious obstacle to admission to the profession even for candidates who have chosen it relatively late. The main factors determining the level of these requirements are the available facilities for preparation, the needs of the pupils, and the economic or salary conditions.

Relatively few young people deliberately choose before entering college the teaching of secondary mathematics as a life work. In the more frequent or more typical case, the college student who will ultimately become a teacher of secondary mathematics makes the choice gradually, perhaps unconsciously, late in the college course or even after its completion, perhaps after some trial of teaching in other fields. The possible supply of young people who have the real desire to become teachers of mathematics is so meager in comparison with the almost unlimited needs of the country that every effort should be made to develop and maintain that desire and all possible encouragement given those who manifest it. If, as will usually be the case, the desire is associated with the necessary mathematical capacity, it will not be wise to hamper the candi-

date by requiring too high attainments, though as a matter of course he will need guidance in continuing his preparation for a profession of exceptional difficulty and exceptional opportunity.

Another factor which must tend to restrict requirements of high mathematical attainment is the importance to the candidate of breadth of preparation. In college he may be in doubt as to becoming a teacher of mathematics or physics or some other subject. It is unwise to hasten the choice. In many cases the secondary teacher must be prepared in more than one field, and to the future teacher of mathematics preparation in physics and drawing, not to mention chemistry, engineering, etc., may be at least as valuable as purely mathematical college electives beyond the calculus.

In the second sense — of standards of scientific attainment to be held by the colleges and normal schools — these institutions should make every effort

1. To awaken interest in the subject and the teaching of it in as many young people of the right sort as possible.
2. To give them the best possible opportunity for professional preparation and improvement, both before and after the beginning of teaching.

How the matter of requirements for appointment will actually work out in a given community will inevitably depend upon conditions of time and place, varying widely in character and degree. In many communities it is already practicable and customary to require not less than two years of college work in mathematics, including elementary calculus, with provision for additional electives. Such a requirement the committee would strongly recom-

mend, recognizing, however, that in some localities it would be for the present too restrictive of the supply. In some cases preparation in the pedagogy, philosophy, and history of mathematics could be reasonably demanded or at least given weight; in other cases, any considerable time spent upon them would be of doubtful value. In all cases requirements should be carefully adjusted to local conditions with a view to recognizing the value both of broad and thorough training on the part of those entering the profession and of continued preparation and improvement by summer work and the like. Particular pains should be taken that such preparation is made accessible and attractive in the colleges and normal schools from which teachers are drawn.

It is naturally important that entrance to the profession should not be much delayed by needlessly high or extended requirements, and the danger of creating a teacher who may be too much a specialist for school work and too little for college teaching must be guarded against. There may naturally also be a wide difference between requirements in a strong school offering many electives and a weaker one or a junior high school. Practically, it may be fair to expect that the stronger schools will maintain their standards not by arbitrary or general requirements for entrance to the profession but often by recruiting from other schools teachers who have both high attainments and successful teaching experience.

Programs of courses for colleges and normal schools preparing teachers in secondary mathematics will be found in Chapter XIV, together with an account of existing conditions.[13]

[[13] Not included in the present edition. But see extract, pp. 126 f, 130 f.]

III

MATHEMATICS FOR YEARS SEVEN, EIGHT AND NINE

I. Introduction

There is a well-marked tendency among school administrators to consider grades one to six, inclusive, as constituting the elementary school and to consider the secondary school period as commencing with the seventh grade and extending through the twelfth.[1] Conforming to this view, the content of the courses of study in mathematics for grades seven, eight, and nine are considered together. In the succeeding chapter the content for grades ten, eleven, and twelve is considered.

The committee is fully aware of the widespread desire on the part of teachers throughout the country for a detailed syllabus by years or half-years which shall give the best order of topics with specific time allotments for each. This desire cannot be met at the present time for the simple reason that no one knows what is the best order of topics, nor how much time should be devoted to each in an ideal course. The committee feels that its recommendations should be so formulated as to give every encour-

[1] "We therefore recommend a reorganization of the school system whereby the first six years shall be devoted to elementary education designed to meet the needs of pupils of approximately 6 to 12 years of age; and the second six years to secondary education designed to meet the needs of approximately 12 to 18 years of age. . . . The six years to be devoted to secondary education may well be divided into two periods, which may be designated as the junior and senior periods." *Cardinal Principles of Secondary Education*, U.S. Bureau of Education, Bulletin no. 55, 1918, p. 18.

agement to further experimentation rather than to restrict the teacher's freedom by a standardized syllabus.

However, certain suggestions as to desirable arrangements of the material are offered in a later section (Section III) of this chapter, and in Chapter XII [2] there will be found detailed outlines giving the order of presentation and time allotments in actual operation in schools of various types. This material should be helpful to teachers and administrators in planning courses to fit their individual needs and conditions.

It is the opinion of the committee that the material included in this chapter should be required of all pupils. It includes mathematical knowledge and training which is likely to be needed by every citizen. Differentiation due to special needs should be made after and not before the completion of such a general minimum foundation. Such portions of the recommended content as have not been completed by the end of the ninth year should be required in the following year.

The general principles which have governed the selection of the material presented in the next section and which should govern the point of view of the teaching have already been stated (Chapter II). At this point it seems desirable to recall specifically what was then said concerning principles governing the organization of material, the importance to be attached to the development of insight and understanding and of ability to think clearly in terms of relationships (dependence), and the limitations imposed on drill in algebraic manipulation. In addition we would call attention to the following considerations:

[[2] Not included in the present edition.]

It is assumed that at the end of the sixth school year the pupil will be able to perform with accuracy and with a fair degree of speed the fundamental operations with integers and with common and decimal fractions.[3] The fractions here referred to are such simple ones in common use as are set forth in detail under A(*c*) in the following section. It may be pointed out that the standard of attainment here implied is met in a large number of schools, as is shown by various tests now in use (see Chapter XIII[4]), and can easily be met generally if time is not wasted on the relatively unimportant parts of the subject.

In adapting instruction in mathematics to the mental traits of pupils care should be taken to maintain the mental growth too often stunted by secondary school materials and methods, and an effort should be made to associate with inquisitiveness, the desire to experiment, the wish to know "how and why" and the like, the satisfaction of these needs.

In the years under consideration it is also especially important to give the pupils as broad an outlook over the various fields of mathematics as is consistent with sound scholarship. These years especially are the ones in which the pupil should have the opportunity to find himself, to test his abilities and aptitudes, and to secure information and experience which will help him choose wisely his later courses and ultimately his life work.

[[3] In most progressive schools the study of decimal fractions includes an introduction to percentage, per cent being looked upon simply as another type of decimal.]

[[4] Not included in the present edition. But see statement prepared by Professor C. B. Upton on pp. 133–137.]

II. Material for Grades Seven, Eight, and Nine

In the material outlined in the following pages no attempt is made to indicate the most desirable order of presentation. Stated by topics rather than years the mathematics of grades seven, eight, and nine may properly be expected to include the following:

A. Arithmetic. (*a*) The fundamental operations of arithmetic.

(*b*) Tables of weights and measures in general practical use, including the most common metric units (meter, centimeter, millimeter, kilometer, gram, kilogram, liter). The meaning of such foreign monetary units as pound, franc, and mark.

(*c*) Such simple fractions as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{1}{8}$; others than these to have less attention.

(*d*) Facility and accuracy in the four fundamental operations; time tests, taking care to avoid subordinating the teaching to the tests or to use the tests as measures of the teacher's efficiency. (See Chapter XIII.[5])

(*e*) Such simple short cuts in multiplication and division as that of replacing multiplication by 25 by multiplying by 100 and dividing by 4.

(*f*) Percentage. Interchanging common fractions and per cents; finding any per cent of a number; finding what per cent one number is of another; finding a number when a certain per cent of it is known; and such applications of percentage as come within the student's experience.

[[5] Not included in the present edition. But see the statement by Professor C. B. Upton especially prepared for this edition, on pp. 133–137. It has been suggested that it would be a step in advance if the case of division of one fraction by another received but little attention, representing as it does an operation rarely needed except in a limited number of technical lines of work.]

(*g*) Line, bar, and circle graphs, wherever they can be used to advantage.

(*h*) Arithmetic of the home: household accounts, thrift, simple bookkeeping, methods of sending money, parcel post.

Arithmetic of the community: property and personal insurance, taxes.

Arithmetic of banking: savings accounts, checking accounts.

Arithmetic of investment: real estate, elementary notions of stocks and bonds, postal savings.

(*i*) Statistics: fundamental concepts, statistical tables and graphs; pictograms; graphs showing simple frequency distributions.

It will be seen that the material listed above includes some material of earlier instruction. This does not mean that this material is to be made the direct object of study but that drill in it shall be given in connection with the new work. It is felt that this shift in emphasis will make the arithmetic processes here involved much more effective and will also result in a great saving of time.

The amount of time devoted to arithmetic as a distinct subject should be greatly reduced from what is at present customary. This does not mean a lessening of emphasis on drill in arithmetic processes for the purpose of securing accuracy and speed. The need for continued arithmetic work and numerical computation throughout the secondary school period is recognized elsewhere in this report. (Chapter II.)

The applications of arithmetic to business should be continued late enough in the course to bring to their study

the pupil's greatest maturity, experience, and mathematical knowledge, and to insure real significance of this study in the business and industrial life which many of the pupils will enter at the close of the eighth or ninth school year. (See I, below.) In this connection care should be taken that the business practices taught in the schools are in accord with the best actual usage. Arithmetic should not be completed before the pupil has acquired the power of using algebra as an aid.

B. Intuitive geometry. (*a*) The direct measurement of distances and angles by means of a linear scale and protractor. The approximate character of measurement. An understanding of what is meant by the degree of precision as expressed by the number of "significant" figures.

(*b*) Areas of the square, rectangle, parallelogram, triangle, and trapezoid; circumference and area of a circle; surfaces and volumes of solids of corresponding importance; the construction of the corresponding formulas.

(*c*) Practice in numerical computation with due regard to the number of figures used or retained.

(*d*) Indirect measurement by means of drawings to scale; use of square ruled paper.

(*e*) Geometry of appreciation; geometric forms in nature, architecture, manufacture, and industry.

(*f*) Simple geometric constructions with ruler and compasses, T-square, and triangle, such as that of the perpendicular bisector, the bisector of an angle, and parallel lines.

(*g*) Familiarity with such forms as the equilateral triangle, the $30^\circ - 60^\circ$ right triangle, and the isosceles right triangle; symmetry; a knowledge of such facts as those

concerning the sum of the angles of a triangle and the pythagorean relation; simple cases of geometric loci in the plane and in space.

(*h*) Informal introduction to the idea of similarity.

The work in intuitive geometry should make the pupil familiar with the elementary ideas concerning geometric forms in the plane and in space with respect to shape, size, and position. Much opportunity should be provided for exercising space perception and imagination. The simpler geometric ideas and relations in the plane may properly be extended to three dimensions. The work should, moreover, be carefully planned so as to bring out geometric relations and logical connections. Before the end of this intuitive work the pupil should have definitely begun to make inferences and to draw valid conclusions from the relations discovered. In other words, this informal work in geometry should be so organized as to make it a gradual approach to, and provide a foundation for, the subsequent work in demonstrative geometry.

C. Algebra. 1. The formula — its construction, meaning, and use (*a*) as a concise language; (*b*) as a shorthand rule for computation; (*c*) as a general solution; (*d*) as an expression of the dependence of one variable upon another.

The pupil will already have met the formula in connection with intuitive geometry. The work should now include translation from English into algebraic language, and vice versa, and special care should be taken to make sure that the new language is understood and used intelligently. The nature of the dependence of one variable in a formula upon another should be examined and an-

alyzed, with a view to seeing "how the formula works." (See Chapter VII.)

2. Graphs and graphic representations in general — their construction and interpretation in (*a*) representing facts (statistical, etc.); (*b*) representing dependence; (*c*) solving problems.

After the necessary technique has been adequately presented graphic representation should not be considered as a separate topic but should be used throughout, whenever helpful, as an illustrative and interpretative instrument.

3. Positive and negative numbers — their meaning and use: (*a*) as expressing both magnitude and one of two opposite directions or senses; (*b*) their graphic representation; (*c*) the fundamental operations applied to them.

4. The equation — its use in solving problems:

(*a*) Linear equations in one unknown — their solution and applications.

(*b*) Simple cases of quadratic equations when arising in connection with formulas and problems.

(*c*) Equations in two unknowns, with numerous concrete illustrations.

(*d*) Various simple applications of ratio and proportion in cases in which they are generally used in problems of similarity and in other problems of ordinary life. In view of the usefulness of the ideas and training involved, this subject may also properly include simple cases of variation.

5. Algebraic technique: (*a*) The fundamental operations.

Their connection with the rules of arithmetic should be clearly brought out and made to illuminate numerical processes. Drill in these operations should be limited

strictly in accordance with the principle mentioned in Chapter II, page 141. In particular, "nests" of parentheses should be avoided, and multiplication and division should not involve much beyond monomial and binomial multipliers, divisors, and quotients.

(*b*) Factoring. The only cases that need be considered are (i) common factors of the terms of a polynomial; (ii) the difference of two squares; (iii) trinomials of the second degree that can be easily factored by trial.

(*c*) Fractions. Here again the intimate connection with the corresponding processes of arithmetic should be made clear and should serve to illuminate such processes. The four fundamental operations with fractions should be considered only in connection with simple cases and should be applied constantly throughout the course so as to gain the necessary accuracy and facility.

(*d*) Exponents and radicals. The work done on exponents and radicals should be confined to the simplest material required for the treatment of formulas. The laws for positive integral exponents should be included. The consideration of radicals should be confined to transformations of the following types: $\sqrt{a^2b} = a\sqrt{b}$, $\sqrt{a/b} = \frac{1}{b}\sqrt{ab}$ and $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$, and to the numerical evaluation of simple expressions involving the radical sign. A process for finding the square root of a number should be included, but not for finding the square root of a polynomial.

(*e*) Stress should be laid upon the need for checking solutions.

D. Numerical trigonometry. (*a*) Definition of sine, cosine, and tangent.

(*b*) Their elementary properties as functions.

(*c*) Their use in solving problems involving right triangles.

(*d*) The use of tables of these functions (to three or four places).

The introduction of the elementary notions of trigonometry into the earlier courses in mathematics has not been as general in the United States as in foreign countries. (See Chapter XI.[6]) Among the reasons for an early introduction of this topic are these: Its practical usefulness for many citizens; the insight it gives into the nature of mathematical methods, particularly those concerned with indirect measurement, and into the rôle that mathematics plays in the life of the world; the fact that it is not difficult and that it offers wide opportunity for concrete and significant application; and the interest it arouses in the pupils. It should be based upon the work in intuitive geometry, with which it has intimate contacts (see B, (*d*), (*h*), above), and should be confined to the simplest material needed for the numerical treatment of the problems indicated. Relations between the trigonometric functions need not be considered.

E. Demonstrative geometry. The demonstration of a limited number of propositions, with no attempt to limit the number of fundamental assumptions, the principal purpose being to show to the pupil what "demonstration" means.

Many of the geometric facts previously inferred intuitively may be used as the basis upon which the demonstrative work is built. This is not intended to preclude the possibility of giving at a later time rigorous proofs

[[6] Not included in the present edition.]

of some of the facts inferred intuitionally. It should be noted that from the strictly logical point of view the attempt to reduce to a minimum the list of axioms, postulates, or assumptions is not at all necessary, and from a pedagogical point of view such an attempt in an elementary course is very undesirable. It is necessary, however, that those propositions which are to be used as the basis of subsequent formal proofs be explicitly listed and their logical significance recognized.

In regard to demonstrative geometry some teachers have objected to the introduction of such work below the tenth grade on the ground that with such immature pupils as are found in the ninth grade nothing worth while could be accomplished in the limited time available. These teachers may be right with regard to conditions prevailing or likely to prevail in the majority of schools in the immediate future. The committee has therefore in a later section of this chapter (Section III) made alternative provision for the omission of work in demonstrative geometry.

On the other hand, it is proper to call attention to the fact that certain teachers have successfully introduced a limited amount of work in demonstrative geometry into the ninth grade (see Chapter XII[7]) and that it would seem desirable that others should make the experiment when conditions are favorable. Much of the opposition is probably due to a failure to realize the extent to which the work in intuitive geometry, if properly organized, will prepare the way for the more formal treatment, and to a misconception of the purposes and extent of the work in demonstrative geometry that is proposed. In reaching a

[7 Not included in the present edition.]

decision on this question teachers should keep in mind that it is one of their important duties and obligations, in the grades under consideration, to show their pupils the nature, content, and possibilities of later courses in their subject and to give to each pupil an opportunity to determine his aptitudes and preferences therefor. The omission in the earlier courses of all work of a demonstrative nature in geometry would disregard one educationally important aspect of mathematics.

F. History and biography. Teachers are advised to make themselves reasonably acquainted with the leading events in the history of mathematics, and thus to know that mathematics has developed in answer to human needs, intellectual as well as technical. They should use this material incidentally throughout their courses for the purpose of adding to the interest of the pupils by means of informal talks on the growth of mathematics and on the lives of the great makers of the science.

G. Optional topics. Certain schools have been able to cover satisfactorily the work suggested in sections A to F before the end of the ninth grade (see Chapter XII[8]). The committee looks with favor on the efforts, in such schools, to introduce earlier than is now customary certain topics and processes which are closely related to modern needs, such as the meaning and use of fractional and negative exponents, the use of the slide-rule, the use of logarithms and of other simple tables, and simple work in arithmetic and geometric progressions, with modern applications to such financial topics as interest and annuities and to such scientific topics as falling bodies and laws of growth.

[[8] Not included in the present edition.]

H. Topics to be omitted or postponed. In addition to the large amount of drill in algebraic technique already referred to, the following topics should, in accordance with our basic principles, be excluded from the work of grades seven, eight, and nine; some of them will properly be included in later courses (see Chapter IV):

Highest common factor and lowest common multiple, except the simplest cases involved in the addition of simple fractions.

The theorems on proportion relating to alternation, inversion, composition, and division.

Literal equations, except such as appear in common formulas, including the derivation of formulas and of geometric relations, or to show how needless computation may be avoided.

Radicals, except as indicated in a previous section.

Square root of polynomials.

Cube root.

Theory of exponents.

Simultaneous equations in more than two unknowns.

The binomial theorem.

Imaginary and complex numbers.

Radical equations except such as arise in dealing with elementary formulas.

I. Problems. As already indicated, much of the emphasis now generally placed on the formal exercise should be shifted to the "concrete" or "verbal" problem. The selection of problem material is, therefore, of the highest importance.

The demand for "practical" problems should be fully met in so far as the maturity and previous experience of the pupil will permit. But above all, the problems must

be "real" *to the pupil,* must connect with his ordinary thought, and must be within the world of his experience and interest. "The educational utility of problems is not to be measured by their commercial or scientific value, but by their degree of reality for the pupils. . . . They must exemplify those leading ideas which it is desired to impart, and they must do so through media which are real to those under instruction. The reality is found in the students, the utility in their acquisition of principles." [9]

There should be, moreover, a conscious effort through the selection of problems to correlate the work in mathematics with the other courses of the curriculum, especially in connection with courses in science. The introduction of courses in "general science" increases the opportunities in this direction.

J. Numerical computation, use of tables, etc. The solution of problems should offer opportunity throughout the grades under consideration for considerable arithmetical and computational work. In this connection attention should be called to the importance of exercising common sense and judgment in the use of approximate data, keeping in mind the fact that all data secured from measurement are approximate. A pupil should be led to see the absurdity of giving the area of a circle to a thousandth of a square inch when the radius has been measured only to the nearest inch. He should understand the conception of "the number of significant figures" and should not retain more figures in his result than are warranted by the accuracy of his data.[10] The ideals of ac-

[9] Carson: *Mathematical Education,* pp. 42–50.

[[10] In view of this fact it is unfortunate that exercises in the addition and subtraction of decimals should ever involve cases with varying numbers

curacy and of self-reliance and the necessity of checking all numerical results should be emphasized. An insight into the nature of tables, including some elementary notions as to interpolation, is highly desirable. The use of tables of various kinds (such as squares and square roots, interest, and trigonometric functions) to facilitate computation and to develop the idea of dependence should be encouraged.

III. Suggested Arrangements of Material

In approaching the problem of arranging or organizing this material it is necessary to consider the different situations that may have to be met.

1. **The junior high school.** In view of the fact that under this form of school organization pupils may be expected to remain in school until the end of the junior high school period instead of leaving in large numbers at the end of the eighth school year, the mathematics of the three years of the junior high school should be planned as a unit, and should include the material recommended in the preceding section. There remains the question as to the order in which the various topics should be presented and the amount of time to be devoted to each. The committee has already stated its reasons for not attempting to answer this question (see Section I of this chapter). The following plans for the distribution of time are, however, suggested in the hope that they may be helpful; but no one of them is recommended as superior to the others, and only the large divisions of material are mentioned.

of decimal places. Since all such work results from measurements of some kind, and since in any given problem those measurements should always be made to same degree of accuracy, there is no reality in these operations unless the numbers have the same number of decimal places.]

YEARS SEVEN, EIGHT, AND NINE

PLAN A

First year: Applications of arithmetic, particularly in such lines as relate to the home, to thrift, and the the various school subjects; intuitive geometry.

Second year: Algebra; applied arithmetic, particularly in such lines as relate to commercial, industrial and social needs.

Third year: Algebra, trigonometry, demonstrative geometry.[11]

By this plan the demonstrative geometry is introduced in the third year, and arithmetic is practically completed in the second year.

PLAN B

First year: Applied arithmetic (as in plan A); intuitive geometry.

Second year: Algebra, intuitive geometry, trigonometry.

Third year: Applied arithmetic, algebra, trigonometry, demonstrative geometry.[11]

By this plan trigonometry is taken up in two years, and the arithmetic is transferred from the second year to the third year.

PLAN C

First year: Applied arithmetic (as in plan A), intuitive geometry, algebra.

Second year: Algebra, intuitive geometry.

Third year: Trigonometry, demonstrative geometry,[11] applied arithmetic.

By this plan Algebra is confined chiefly to the first two years.

PLAN D

First year: Applied arithmetic (as in plan A), intuitive geometry.

[[11] It is to be remembered that in all of these plans, "demonstrative geometry" means merely a brief introduction, "the principal purpose being to show the pupil what 'demonstration' means." See p. 35, E.]

Second year: Intuitive geometry, algebra.
Third year: Algebra, trigonometry, applied arithmetic.
By this plan demonstrative geometry is omitted entirely.

PLAN E

First year: Intuitive geometry, simple formulas, elementary principles of statistics, arithmetic (as in plan A).
Second year: Intuitive geometry, algebra, arithmetic.
Third year: Geometry, numerical trigonometry, arithmetic.

2. Schools organized on the 8-4 plan. It cannot be too strongly emphasized that, in the case of the older and at present more prevalent plan of the 8–4 school organization, the work in mathematics of the seventh, eighth, and ninth grades should also be organized to include the material here suggested.

The prevailing practice of devoting the seventh and eighth grades almost exclusively to the study of arithmetic is generally recognized as a wasteful marking of time. It is mainly in these years that American children fall behind their European brothers and sisters. No essentially new arithmetic principles are taught in these years, and the attempt to apply the previously learned principles to new situations in the more advanced business and economic aspects of arithmetic is doomed to failure on account of the fact that the situations in question are not and cannot be made real and significant to pupils of this age. We need only refer to what has already been said in this chapter on the subject of problems (I, Section II).

The same principles should govern the selection and arrangement of material in mathematics for the seventh and eighth grades of a grade school as govern the selec-

tion for the corresponding grades of a junior high school, with this exception: Under the 8–4 form of organization many pupils will leave school at the end of the eighth year. This fact must receive due consideration. The work of the seventh and eighth years should be so planned as to give the pupils in these grades the most valuable mathematical information and training that they are capable of receiving in those years, with little reference to courses that they may take in later years. As to possibilities for arrangement, reference may be made to the plans given above for the first two years of the junior high school. When the work in mathematics of the seventh and eighth grades has been thus reorganized, the work of the first year of a standard four-year high school should complete the program suggested.

Finally, there must be considered the situation in those four-year high schools in which the pupils have not had the benefit of the reorganized instruction recommended for grades seven and eight. It may be hoped that this situation will be only temporary, although it must be recognized that, owing to a variety of possible reasons (lack of adequately prepared teachers in grades seven and eight, lack of suitable textbooks, administrative inertia, and the like), the new plans will not be immediately adopted and that, therefore, for some years many high schools will have to face the situation implied.

In planning the work of the ninth grade under these conditions teachers and administrative officers should again be guided by the principle of giving the pupils the most valuable mathematical information and training which they are capable of receiving in this year, with little reference to future courses which the pupil may or

may not take. It is to be assumed that the work of this year is to be required of all pupils. Since for many this will constitute the last of their mathematical instruction, it should be so planned as to give them the widest outlook consistent with sound scholarship.

Under these conditions it would seem desirable that the work of the ninth grade should contain both algebra and geometry. It is, therefore, recommended that about two thirds of the time be devoted to the most useful parts of algebra, including the work on numerical trigonometry, and that about one third of the time be devoted to geometry, including the necessary informal introduction and, if feasible, the first part of demonstrative geometry.

It should be clear that owing to the greater maturity of the pupils much less time need be devoted in the ninth grade to certain topics of intuitive geometry (such as direct measurement, for example) than is desirable when dealing with children in earlier grades. Even under the conditions presupposed, pupils will be acquainted with most of the fundamental geometric forms and with the mensuration of the most important plane and solid figures. The work in geometry in the ninth grade can then properly be made to center about indirect measurement and the idea of similarity (leading to the processes of numerical trigonometry), and such geometric relations as the sum of the angles of a triangle, the pythagorean proposition, congruence of triangles, parallel and perpendicular lines, quadrilaterals, and the more important simple constructions.

IV

MATHEMATICS FOR YEARS TEN, ELEVEN AND TWELVE

I. Introduction

The committee has in the preceding chapter expressed its judgment that the material there recommended for the seventh, eighth, and ninth years should be required of all pupils. In the tenth, eleventh, and twelfth years, however, the extent to which election of subjects is permitted will depend on so many factors of a general character that it seems unnecessary and inexpedient for the present committee to urge a positive requirement beyond the minimum one already referred to. The subject must, like others, stand or fall on its intrinsic merit or on the estimate of such merit by the authorities responsible at a given time and place. The committee believes nevertheless that every standard high school should not merely offer courses in mathematics for the tenth, eleventh, and twelfth years, but should encourage a large proportion of its pupils to take them. Apart from the intrinsic interest and great educational value of the study of mathematics, it will in general be necessary for those preparing to enter college or to engage in the numerous occupations involving the use of mathematics to extend their work beyond the minimum requirement.

The present chapter is intended to suggest for students in general courses the most valuable mathematical training that will appropriately follow the courses outlined in the previous chapter. Under present conditions most

of this work will normally fall in the last three years of the high school; that is, in general, in the tenth, eleventh, and twelfth years.[1]

The selection of material is based on the general principles formulated in Chapter II. At this point attention need be directed only to the following:

1. In the years under consideration it is proper that some attention be paid to the students' vocational or other later educational needs.

2. The material for these years should include as far as possible those mathematical ideas and processes that have the most important applications in the modern world. As a result, certain material will naturally be included that at present is not ordinarily given in secondary school courses; as, for instance, the material concerning the calculus. On the other hand, certain other material that is now included in college entrance requirements will be excluded.[2] The results of an investigation made by the National Committee in connection with a study of these requirements indicate that modifications to meet these changes will be desirable from the standpoint of both college and secondary school (see Chapter V).

3. During the years now under consideration an increasing amount of attention should be paid to the logical organization of the material, with the purpose of developing habits of logical memory, appreciation of logical structure, and ability to organize material effectively.

It cannot be too strongly emphasized that the broadening of content of high school courses in mathematics suggested in the present and in previous chapters will ma-

[[1] These years are coming to be known as the senior high school.]
[[2] "Will be made optional" more accurately expresses the meaning.]

terially increase the usefulness of these courses to those who pursue them. It is of prime importance that educational administrators and others charged with the advising of students should take careful account of this fact in estimating the relative importance of mathematical courses and their alternatives. The number of important applications of mathematics in the activities of the world is to-day very large and is increasing at a very rapid rate. This aspect of the progress of civilization has been noted by all observers who have combined a knowledge of mathematics with an alert interest in the newer developments in other fields. It was revealed in very illuminating fashion during the recent war by the insistent demand for persons with varying degrees of mathematical training for many war activities of the first moment. If the same effort were made in time of peace to secure the highest level of efficiency available for the specific tasks of modern life, the demand for those trained in mathematics would be no less insistent; for it is in no wise true that the applications of mathematics in modern warfare are relatively more important or more numerous than its applications in those fields of human endeavor which are of a constructive nature.

There is another important point to be kept in mind in considering the relative value to the average student of mathematical and various alternative courses. If the student who omits the mathematical courses has need of them later, it is almost invariably more difficult, and it is frequently impossible, for him to obtain the training in which he is deficient. In the case of a considerable number of alternative subjects a proper amount of reading in spare hours at a more mature age will ordinarily furnish

him the approximate equivalent to that which he would have obtained in the way of information in a high school course in the same subject. It is not, however, possible to make up deficiencies in mathematical training in so simple a fashion. It requires systematic work under a competent teacher to master properly the technique of the subject, and any break in the continuity of the work is a handicap for which increased maturity rarely compensates. Moreover, when the individual discovers his need for further mathematical training it is usually difficult for him to take the time from his other activities for systematic work in elementary mathematics.

II. Recommendations for Elective Courses

The following topics are recommended for inclusion in the mathematical electives open to pupils who have satisfactorily completed the work outlined in the preceding chapter, comprising arithmetic, the elementary notions of algebra, intuitive geometry, numerical trigonometry, and a brief introduction to demonstrative geometry.

1. **Plane demonstrative geometry.** The principal purposes of the instruction in this subject are: To exercise further the spatial imagination of the student, to make him familiar with the great basal propositions and their applications, to develop understanding and appreciation of a deductive proof and the ability to use this method of reasoning where it is applicable, and to form habits of precise and succinct statement, of the logical organization of ideas, and of logical memory. Enough time should be spent on this subject to accomplish these purposes.

The following is a suggested list of topics under which

the work in demonstrative geometry may be organized:[3] (*a*) Congruent triangles, perpendicular bisectors, bisectors of angles; (*b*) arcs, angles, and chords in circles; (*c*) parallel lines and related angles, parallelograms; (*d*) the sum of the angles for triangle and polygon; (*e*) secants and tangents to circles with related angles, regular polygons; (*f*) similar triangles, similar figures; (*g*) areas; numerical computation of lengths and areas, based upon geometric theorems already established.

Under these topics constructions, loci, originals and other exercises are to be included.

It is recommended that the formal theory of limits and of incommensurable cases be omitted, but that the ideas of limit and of incommensurable magnitudes receive informal treatment.

It is believed that a more frequent use of the idea of motion in the demonstration of theorems is desirable, both from the point of view of gaining greater insight and of saving time.[4]

If the great basal theorems are selected and effectively organized into a logical system, a considerable reduction (from thirty to forty per cent) can be made in the number of theorems given either in the Harvard list or in the report of the Committee of Fifteen. Such a reduction is exhibited in the lists prepared by the committee and printed later in this report (Chapter VI).[5] In this connection it may be suggested that more attention than is

[3] It is not intended that the order here given should imply anything as to the order of presentation. (See also Chapter VI.)

[4] Reference may here be made to the treatment given in recent French texts such as those by Bourlet and Méray.

[[5] See also the College Entrance Examination Board lists referred to in Note (2), p. 3.]

now customary may profitably be given to those methods of treatment which make consistent use of the idea of motion (already referred to), continuity (the tangent as the limit of a secant, etc.), symmetry, and the dependence of one geometric magnitude upon another.

If the student has had a satisfactory course in intuitive geometry and some work in demonstration before the tenth grade, he may find it possible to cover a minimum course in demonstrative geometry, giving the great basal theorems and constructions, together with exercises, in the ninety periods constituting a half-year's work.[6]

2. **Algebra.** (*a*) *Simple functions of one variable:* Numerous illustrations and problems involving linear, quadratic, and other simple functions including formulas from science and common life. More difficult problems in variation than those included in the earlier course.

(*b*) *Equations in one unknown:* Various methods for solving a quadratic equation (such as factoring, completing the square, use of formula) should be given. In connection with the treatment of the quadratic a very brief discussion of complex numbers should be included. Simple cases of the graphic solution of equations of degree higher than the second should be discussed and applied.

[[6] The editor has not as yet heard of any school that has attempted to cover such a minimum course in half a year. The experience of the last few years has shown conclusively, however, that the work in intuitive geometry of the junior high school makes it already possible to save, on a conservative basis, from six to eight weeks on the traditional year course in plane geometry. It may be expected that this situation will soon lead to the organization of courses covering the fundamental topics of plane and solid geometry in one year, along the lines, perhaps, suggested by the C.E.E.B. definition of "Plane and Solid Geometry, Minor Requirement." (See Document 108, referred to in note (1) p. 3; see also below, p. 167.)]

(*c*) *Equations in two or three unknowns:* The algebraic solution of linear equations in two or three unknowns and the graphic solution of linear equations in two unknowns should be given. The graphic and algebraic solution of a linear and a quadratic equation and of two quadratics that contain no first degree term and no xy term should be included.

(*d*) *Exponents, radicals, and logarithms:* The definitions of negative, zero, and fractional exponents should be given, and it should be made clear that these definitions must be adopted if we wish such exponents to conform to the laws for positive integral exponents. Reduction of radical expressions to those involving fractional exponents should be given as well as the inverse transformation. The rules for performing the fundamental operations on expressions involving radicals, and such transformations as

$$\sqrt[n]{a/b} = \frac{1}{b}\sqrt[n]{ab^{n-1}},\ \sqrt[n]{a^n b} = a\sqrt[n]{b},\ \frac{a}{\sqrt{b}+\sqrt{c}} = \frac{a(\sqrt{b}-\sqrt{c})}{b-c}$$

should be included. In close connection with the work on exponents and radicals there should be given as much of the theory of logarithms as is involved in their application to computation and sufficient practice in their use in computation to impart a fair degree of facility.

(*e*) *Arithmetic and geometric progressions:* The formulas for the nth term and the sum of n terms should be derived and applied to significant problems.

(*f*) *Binomial theorem:* A proof for positive integral exponents should be given; it may also be stated that the formula applies to the case of negative and fractional exponents under suitable restrictions, and the problems

may include the use of the formula in these cases as well as in the case of positive integral exponents.

3. **Solid geometry.** The aim of the work in solid geometry should be to exercise further the spatial imagination of the student and to give him both a knowledge of the fundamental spatial relationships and the power to work with them. It is felt that the work in plane geometry gives enough training in logical demonstration to warrant a shifting of emphasis in the work on solid geometry away from this aspect of the subject and in the direction of developing greater facility in visualizing spatial relations and figures, in representing such figures on paper, and in solving problems in mensuration.[7]

For many of the practical applications of mathematics it is of fundamental importance to have accurate space perceptions. Hence it would seem wise to have at least some of the work in solid geometry come as early as possible in the mathematical courses, preferably not later than the beginning of the eleventh school year. Some schools will find it possible and desirable to introduce the more elementary notions of solid geometry in connection with related ideas of plane geometry.

The work in solid geometry should include numerous exercises in computation based on the formulas established. This will serve to correlate the work with arithmetic and algebra and to furnish practice in computation.

The following provisional outline of subject-matter is submitted:

(*a*) Propositions relating to lines and planes, and to dihedral and trihedral angles.

[7 See Note 6, p. 50.]

(*b*) Mensuration of the prism, pyramid, and frustum; the (right circular) cylinder, cone and frustum, based on an informal treatment of limits; the sphere and the spherical triangle.

(*c*) Spherical geometry.

(*d*) Similar solids.

Such theorems as are necessary as a basis for the topics here outlined should be studied in immediate connection with them.

Desirable simplification and generalization may be introduced into the treatment of mensuration theorems by employing such theorems as Cavalieri's and Simpson's, and the Prismoid Formula; but rigorous proofs or derivations of these need not be included.

Beyond the range of the mensuration topics indicated above, it seems preferable to employ the methods of the elementary calculus (see Section 6, below).

It should be possible to complete a minimum course covering the topics outlined above in not more than one third of a year.[8]

The list of propositions in solid geometry given in Chapter VI should be considered in connection with the general principles stated at the beginning of this section. By requiring formal proofs to a more limited extent than has been customary, time will be gained to attain the aims indicated and to extend the range of geometric information of the pupil. Care must be exercised to make sure that the pupil is thoroughly familiar with the facts, with the associated terminology, with all the neces-

[8] See Note 6, p. 50. It may be added that the practice of teaching plane and solid geometry together has for years been in successful operation in many European schools.

sary formulas, and that he secures the necessary practice in working with and applying the information acquired to concrete problems.

4. **Trigonometry.** The work in elementary trigonometry begun in the earlier years should be completed by including the logarithmic solution of right and oblique triangles, radian measure, graphs of trigonometric functions, the derivation of the fundamental relations between the functions and their use in proving identities and in solving easy trigonometric equations. The use of the transit in connection with the simpler operations of surveying and of the sextant for some of the simpler astronomical observations, such as those involved in finding local time, is of value; but when no transit or sextant is available, simple apparatus for measuring angles roughly may and should be improvised. Drawings to scale should form an essential part of the numerical work in trigonometry. The use of the slide-rule in computations requiring only three-place accuracy and in checking other computations is also recommended.

5. **Elementary statistics.** Continuation of the earlier work to include the meaning and use of fundamental concepts and simple frequency distributions with graphic representations of various kinds and measures of central tendency (average, mode, and median).

6. **Elementary calculus.** The work should include:

(*a*) The general notion of a derivative as a limit indispensable for the accurate expression of such fundamental quantities as velocity of a moving body or slope of a curve.

(*b*) Applications of derivatives to easy problems in rates and in maxima and minima.

(*c*) Simple cases of inverse problems; e.g., finding distance from velocity, etc.

(*d*) Approximate methods of summation leading up to integration as a powerful method of summation.

(*e*) Applications to simple cases of motion, area, volume, and pressure.

Work in the calculus should be largely graphic and may be closely related to that in physics; the necessary technique should be reduced to a minimum by basing it wholly or mainly on algebraic polynomials. No formal study of analytic geometry need be presupposed beyond the plotting of simple graphs.

It is important to bear in mind that, while the elementary calculus is sufficiently easy, interesting, and valuable to justify its introduction, special pains should be taken to guard against any lack of thoroughness in the fundamentals of algebra and geometry; no possible gain could compensate for a real sacrifice of such thoroughness.

It should also be borne in mind that the suggestion of including elementary calculus is not intended for all schools nor for all teachers or all pupils in any school. It is not intended to connect in any direct way with college entrance requirements. The future college student will have ample opportunity for calculus later. The capable boy or girl who is not to have the college work ought not on that account to be prevented from learning something of the use of this powerful tool. The applications of elementary calculus to simple concrete problems are far more abundant and more interesting than those of algebra. The necessary technique is extremely simple. The subject is commonly taught in secondary schools in

England, France, and Germany, and appropriate English texts are available.[9] [10]

7. **History and biography.** Historical and biographical material should be used throughout to make the work more interesting and significant.

8. **Additional Electives.** Additional electives such as *mathematics of investment*, *shop mathematics*, *surveying and navigation*, *descriptive* or *projective geometry* will appropriately be offered by schools which have special needs or conditions, but it seems unwise for the National Committee to attempt to define them pending the results of further experience on the part of these schools.

III. Plans for Arrangement of the Material

In the majority of high schools at the present time the topics suggested can probably be given most advantageously as separate units of a three-year program. However, the National Committee is of the opinion that methods of organization are being experimentally perfected whereby teachers will be enabled to present much of this material more effectively in combined courses unified by one or more of such central ideas as functionality and graphic representation.

As to the arrangement of the material the committee gives below four plans which may be suggestive and helpful to teachers in arranging their courses. No one of them is, however, recommended as superior to the others.

[9] Quotations and typical problems from one of these texts will be found in a supplementary note appended to this chapter.

[[10] See statement by Miss Blair, p. 137.]

YEARS TEN, ELEVEN, AND TWELVE

PLAN A

Tenth year: Plane demonstrative geometry, algebra.
Eleventh year: Statistics, trigonometry, solid geometry.
Twelfth year: The calculus, other elective.

PLAN B

Tenth year: Plane demonstrative geometry, solid geometry.
Eleventh year: Algebra, trigonometry, statistics.
Twelfth year: The calculus, other elective.

PLAN C

Tenth year: Plane demonstrative geometry, trigonometry.
Eleventh year: Solid geometry, algebra, statistics.
Twelfth year: The calculus, other elective.

PLAN D

Tenth year: Algebra, statistics, trigonometry.
Eleventh year: Plane and solid geometry.
Twelfth year: The calculus, other elective.

Additional information on ways of organizing this material will be found in Chapter XII.[11]

Supplementary Note on the Calculus as a High School Subject

In connection with the recommendations concerning the calculus, such questions as the following may arise: Why should a college subject like this be added to a high school program? How can it be expected that high school teachers will have the necessary training and attainments for teaching it? Will not the attempt to teach such a subject result in loss of thoroughness in earlier work? Will anything be gained beyond a mere smattering of the theory? Will the boy or

[[11] Not included in the present edition.]

girl ever use the information or training secured? The subsequent remarks are intended to answer such objections as these and to develop more fully the point of view of the committee in recommending the inclusion of elementary work in the calculus in the high school program.

By the calculus we mean for the present purpose a study of *rates of change*. In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a changing quantity become largest or smallest? How can rates of changing quantities be compared?

These are some of the questions which lead us to study the elementary calculus. Without its essential principles these questions cannot be answered with definiteness.

The following are a few of the specific replies that might be given in answer to the questions listed at the beginning of this note: The difficulties of the college calculus lie mainly outside the boundaries of the proposed work. The elements of the subject present less difficulty than many topics now offered in advanced algebra. It is not implied that in the near future many secondary school teachers will have any occasion to teach the elementary calculus. It is the culminating subject in a series which only relatively strong schools will complete and only then for a selected group of students. In such schools there should always be teachers competent to teach the elementary calculus here intended. No superficial study of calculus should be regarded as justifying any substantial sacrifice of thoroughness. In the judgment of the committee the introduction of elementary calculus necessarily includes sufficient algebra and geometry to compensate for whatever diversion of time from these subjects would be implied.

The calculus of the algebraic polynomial is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, and of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of

mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations — and such training is of value to every one.

The following typical extracts from an English text intended for use in secondary schools may be quoted:

"It has been said that the calculus is that branch of mathematics which schoolboys understand and senior wranglers fail to comprehend. . . . So long as the graphic treatment and practical applications of the calculus are kept in view, the subject is an extremely easy and attractive one. Boys can be taught the subject early in their mathematical career, and there is no part of their mathematical training that they enjoy better or which opens up to them wider fields of useful exploration. . . . The phenomena must first be known practically and then studied philosophically. To reverse the order of these processes is impossible."

The text in question, after an interesting historical sketch, deals with such problems as the following:

A train is going at the rate of 40 miles an hour. Represent this graphically.

At what rate is the length of the daylight increasing or decreasing on December 31, March 26, etc.? (From tabular data.)

A cart going at the rate of 5 miles per hour passes a milestone, and 14 minutes afterwards a bicycle, going in the same direction at 12 miles an hour, passes the same milestone. Find when and where the bicycle will overtake the cart.

A man has 4 miles of fencing wire and wishes to fence in a rectangular piece of prairie land through which a straight river flows, the bank of the stream being utilized as one side of the enclosure. How can he do this so as to inclose as much land as possible?

A circular tin canister closed at both ends has a surface area

of 100 square centimeters. Find the greatest volume it can contain.

Post-office regulations prescribe that the combined length and girth of a parcel must not exceed 6 feet. Find the maximum volume of a parcel whose shape is a prism with the ends square.

A pulley is fixed 15 feet above the ground, over which passes a rope 30 feet long with one end attached to a weight which can hang freely, and the other end is held by a man at a height of 3 feet from the ground. The man walks horizontally away from beneath the pulley at the rate of 3 feet per second. Find the rate at which the weight rises when it is 10 feet above the ground.

The pressure on the surface of a lake due to the atmosphere is known to be 14 pounds per square inch. The pressure in the liquid x inches below the surface is known to be given by the law $dp/dx = 0.036$. Find the pressure in the liquid at a depth of 10 feet.

The arch of a bridge is parabolic in form. It is 5 feet wide at the base and 5 feet high. Find the volume of water that passes through per second in a flood when the water is rushing at the rate of 10 feet per second.

A force of 20 tons compresses the spring buffer of a railway stop through 1 inch, and the force is always proportional to the compression produced. Find the work done by a train which compresses a pair of such stops through 6 inches.

These may illustrate the aims and point of view of the proposed work. It will be noted that not all of them involve calculus, but those that do not lead up to it.

V

COLLEGE ENTRANCE REQUIREMENTS

THE present chapter is concerned with a study of topics and training in elementary mathematics that will have most value as preparation for college work, and with recommendations of definitions of college entrance requirements in elementary algebra and plane geometry.

General considerations. The primary purpose of college entrance requirements is to test the candidate's ability to benefit by college instruction. This ability depends, so far as our present inquiry is concerned, upon (1) general intelligence, intellectual maturity, and mental power; (2) specific knowledge and training required as preparation for the various courses of the college curriculum.

Mathematical ability appears to be a sufficient but not a necessary condition for general intelligence.[1] For this, as well as for other reasons, it would appear that *college entrance requirements in mathematics should be formulated primarily on the basis of the special knowledge and training required for the successful study of courses which the student will take in college.*

The separation of prospective college students from the others in the early years of the secondary school is neither feasible nor desirable. It is therefore obvious that secondary school courses in mathematics cannot be planned

[1] A recent investigation made by the department of psychology at Dartmouth College showed that all students of high rank in mathematics had a high rating on general intelligence; the converse was not true, however.

with specific reference to college entrance requirements. Fortunately there appears to be no real conflict of interest between those students who ultimately go to college and those who do not, so far as mathematics is concerned. It will be made clear in what follows that a course in this subject, covering from two to two and one half years in a standard four-year high school, and so planned as to give the most valuable mathematical training which the student is capable of receiving, will provide adequate preparation for college work.

Topics to be included in high-school courses. In the selection of material of instruction for high-school courses in mathematics, its value as preparation for college courses in mathematics need not be specifically considered. Not all college students study mathematics; it is therefore reasonable to expect college departments in this subject to adjust themselves to the previous preparation of their students. Nearly all college students do, however, study one or more of the physical sciences (astronomy, physics, chemistry) and one or more of the social sciences (history, economics, political science, sociology). Entrance requirements must therefore insure adequate mathematical preparation in these subjects. Moreover, it may be assumed that adequate preparation for these two groups of subjects will be sufficient for all other subjects for which the secondary schools may be expected to furnish the mathematical prerequisites.

The National Committee recently conducted an investigation for the purpose of securing information as to the content of high-school courses of instruction most desirable from the point of view of preparation for college work. A number of college teachers, prominent in their

respective fields, were asked to assign to each of the topics in the following table an estimate of its value as preparation for the elementary courses in their respective subjects. Table 1 gives a summary of the replies, arranged in two groups — "Physical sciences," including astronomy, physics, and chemistry; and "Social sciences," including history, economics, sociology, and political science.

The high value attached to the following topics is significant: Simple formulas — their meaning and use; the linear and quadratic functions and variation; numerical trigonometry; the use of logarithms and other topics relating to numerical computation; statistics. These all stand well above such standard requirements as arithmetic and geometric progression, binomial theorem, theory of exponents, simultaneous equations involving one or two quadratic equations, and literal equations.

These results would seem to indicate that a modification of present college entrance requirements in mathematics is desirable from the point of view of college teachers in departments other than mathematics. It is interesting to note how closely the modifications suggested by this inquiry correspond to the modifications in secondary school mathematics foreshadowed by the study of needs of the high school pupil irrespective of his possible future college attendance. The recommendations made in Chapter II that functional relationship be made the "underlying principle of the course," that the meaning and use of simple formulas be emphasized, that more attention be given to numerical computation (especially to the methods relating to approximate data), and that work on numerical trigonometry and statistics be in-

Table I. — Value of Topics as Preparation for Elementary College Courses

[In the headings of the table, E = essential, C = of considerable value, S = of some value, O = of little or no value, N = number of replies received. The figures in the first four columns of each group are percentages of the number of replies received.]

	Physical Sciences					Social Sciences				
	E	C	S	O	N	E	C	S	O	N
Negative numbers — their meaning and use	79	5	10	5	39	45	17	22	17	18
Imaginary numbers — their meaning and use	23	21	25	31	39	13	13	37	37	16
Simple formulas — their meaning and use	93	5	2	—	41	47	26	21	5	19
Graphic representation of statistical data	57	25	15	3	40	57	24	14	5	21
Graphs (mathematical and empirical):										
(*a*) As a method of representing dependence	62	16	22	—	37	15	54	15	15	13
(*b*) As a method of solving problems	45	20	28	6	25	18	18	46	18	11
The linear function, $y = mx + b$	78	14	8	—	37	29	29	14	29	14
The quadratic function, $y = ax^2 + bx + c$	59	21	17	3	34	8	8	33	50	12
Equations: Problems leading to —										
Linear equations in one unknown	98	2	—	—	41	40	7	20	33	15
Quadratic equations in one unknown	78	15	5	2	40	31	8	8	54	13
Simultaneous linear equations in 2 unknowns	71	24	3	3	38	33	8	—	58	12
Simultaneous linear equations in more than 2 unknowns	43	29	23	6	35	8	8	17	67	12
One quadratic and one linear equation in 2 unknowns	40	24	27	9	33	—	9	9	82	11
Two quadratic equations in 2 unknowns	31	19	28	22	32	—	9	—	91	11
Equations of higher degree than the second	10	32	32	26	31	—	—	9	91	11
Literal equations (other than formulas)	43	18	32	7	28	—	10	40	50	10
Ratio and proportion	84	8	3	5	39	37	26	32	5	19
Variation	50	13	20	17	30	17	33	25	25	12
Numerical computation:										
With approximate data — rational use of significant figures	61	36	—	3	39	40	27	20	13	12
Short-cut methods	27	38	24	10	37	29	35	23	12	17
Use of logarithms	62	29	7	2	42	12	29	29	29	17
Use of other tables to facilitate computation	24	45	26	5	38	18	29	41	12	17
Use of slide-rule	24	39	26	12	38	11	39	28	22	18
Theory of exponents	36	31	25	8	36	—	21	21	57	14
Theory of logarithms	34	26	21	18	38	7	13	20	60	15
Arithmetic progression	16	32	38	13	37	23	29	12	35	17
Geometric progression	19	27	40	14	37	23	25	18	35	17
Binomial theorem	35	32	18	13	37	13	20	27	40	15
Probability	9	32	41	19	32	20	35	35	10	20
Statistics:										
Meaning and use of elementary concepts	23	28	31	17	29	55	36	5	5	22
Frequency distributions and frequency curves	15	19	35	32	26	47	33	10	10	21
Correlation	11	18	39	32	28	33	47	14	5	21
Numerical trigonometry:										
Use of sine, cosine, and tangent in the solution of simple problems involving right triangles	68	21	3	8	38	—	—	25	75	12
Demonstrative geometry	68	15	12	6	34	—	21	43	36	14
Plane trigonometry (usual course)	57	27	11	5	37	8	23	31	38	13
Analytic geometry:										
Fundamental conceptions and methods in the plane	32	45	19	3	31	—	15	38	46	13
Systematic treatment of —										
Straight line	34	37	20	9	35	9	9	18	64	11
Circle	29	43	20	9	35	—	18	9	73	11
Conic sections	18	41	26	15	34	—	9	18	73	11
Polar coordinates	18	26	41	15	34	—	—	18	82	11
Empirical curves and fitting curves to observations	12	38	38	12	34	8	—	25	67	12

TABLE 2. — TOPICS IN ORDER OF VALUE AS PREPARATION FOR ELEMENTARY COLLEGE COURSES

[The figures in the column headed "E" are taken from Table 1, taking in each case the higher of the two "E" ratings there given. The column headed "E + C" gives in each case the sum of the two ratings for "E" and "C." An asterisk indicates that the topic in question is now included in the definitions of the College Entrance Examination Board.[a]]

	E	E+C
*Linear equations in one unknown	98	100
Simple formulas — their meaning and use	93	98
*Ratio and proportion	84	92
*Negative numbers — their meaning and use	79	84
*Quadratic equations in one unknown	78	93
The linear function: $y = mx + b$	78	92
*Simultaneous linear equations in two unknowns	71	95
Numerical trigonometry — the use of the sine, cosine, and tangent in the solution of simple problems involving right triangles	68	89
*Demonstrative geometry	68	83
Use of logarithms in computation	62	91
*Graphs as a method of representing dependence	62	78
Computation with approximate data — rational use of significant figures	61	97
The quadratic function: $y = ax^2 + bx + c$	59	80
Plane trigonometry — usual course	57	84
Graphic representation of statistical data	57	82
Statistics — meaning and use of elementary concepts	55	91
Variation	50	63
Statistics — frequency distributions and curves	47	80
Graphic solution of problems	45	65
*Literal equations	43	61
*Simultaneous linear equations in more than 2 unknowns	43	72
*Simultaneous equations, one quadratic, one linear	40	64
*Theory of exponents	36	67
*Binomial theorem	35	67
Analytic geometry of the straight line	34	71
Theory of logarithms	34	60
Statistics — correlation	33	80
Analytic geometry — Fundamental Conceptions	32	77
*Simultaneous quadratic equations	31	50
Analytic geometry of the circle	29	72
Short-cut methods of computation	29	65
Use of tables in computation (other than logarithms)	24	69
Use of slide rule	24	63
*Arithmetic progression	23	52
*Geometric progression	23	48
Imaginary numbers	23	44
Probability	20	55
Conic sections	18	59
Polar coördinates	18	44
Empirical curves and fitting curves to observations	12	50
Equations of higher degree than the second	10	42

[a] The list includes all the requirements of the college entrance examination board except those relating to algebraic technique. The topic of "Negative numbers" has also been given an asterisk, as it is clearly implied, though not explicitly mentioned in the C.E.E.B. definitions.

[It should be noted that the asterisk refers to topics included in the C.E.E.B. definitions in force in 1923; — the present (1927) requirements include several topics not marked with an asterisk in the above list. See the present requirements reprinted on p. 140.]

cluded, have received widespread approval throughout the country. That they should be in such close accord with the desires of college teachers in the fields of physical and social sciences as to entrance requirements is striking. We find here the justification for the belief expressed earlier in this report that there is no real conflict between the needs of students who ultimately go to college and those who do not.

The attitude of the colleges. Mathematical instruction in this country is at present in a period of transition. While a considerable number of our most progressive schools have for several years given courses embodying most of the recommendations contained in Chapters II, III, and IV of the present report, the large majority of schools are still continuing the older types of courses or are only just beginning to introduce modifications. The movement toward reorganization is strong, however, throughout the country, not only in the standard four-year high schools but also in the newer junior high schools.

During this period of transition it should be the policy of the colleges, while exerting a desirable steadying influence, to help the movement toward a sane reorganization. In particular, they should take care not to place obstacles in the way of changes which are clearly in the interest of more effective college preparation, as well as of better general education.

College entrance requirements will continue to exert a powerful influence on secondary school teaching. Unless they reflect the spirit of sound progressive tendencies, they will constitute a serious obstacle.

In the present chapter revised definitions of college-entrance requirements in plane geometry and elementary

algebra are presented. So far as plane geometry is concerned, the problem of definition is comparatively simple. The proposed definition of the requirement in plane geometry does not differ from the one now in effect under the College Entrance Examination Board. A list of propositions and constructions has, however, been prepared and is given in the next chapter for the guidance of teachers and examiners.

In elementary algebra a certain amount of flexibility is obviously necessary both on account of the quantitative differences among colleges and of the special conditions attending a period of transition. The former differences are recognized by the proposal of a minor and a major requirement in elementary algebra. The second of these includes the first and is intended to correspond with the two-unit rating of the C.E.E.B.

In connection with this matter of units, the committee wishes particularly to disclaim any emphasis upon a special number of years or hours. The unit terminology is doubtless too well established to be entirely ignored in formulating college entrance requirements, but the standard definition of unit [3] has never been precise, and will now become much less so with the inclusion of the newer six-year program. A time allotment of 4 or 5 hours per week in the seventh year can certainly not have the same weight as the same number of hours in the twelfth year, and the disparity will vary with different

[3] The following definition, formulated by the National Committee on Standards of Colleges and Secondary Schools, has been given the approval of the C.E.E.B: "A unit represents a year's study in any subject in a secondary school, constituting approximately a quarter of a full year's work. A four-year secondary school curriculum should be regarded as representing not more than sixteen units of work."

subjects. *What is really important is the amount of subject-matter and the quality of work done in it.* The "unit" cannot be anything but a crude approximation to this. The distribution of time in the school program should not be determined by any arbitrary unit scale.

As a further means of securing reasonable flexibility, the committee recommends that for a limited time — say five years — the option be offered between examinations based on the old and on the new definitions, so far as differences between them may make this desirable.

In view of the changes taking place at the present time in mathematical courses in secondary schools, and the fact that college entrance requirements should so soon as possible reflect desirable changes and assist in their adoption, the National Committee recommends that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on college entrance requirements in mathematics, such a committee to work in close coöperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools.[4]

Proposed Definition of College Entrance Requirements[5]

ELEMENTARY ALGEBRA

Minor requirement. The meaning, use, and evaluation (including the necessary transformations) of simple

[[4] Such a committee has not been appointed. It seems to be the general opinion that such a committee is at present unnecessary.]

[[5] For comparison see the definitions of the C.E.E.B. now in force, which will be found on p. 140.]

formulas involving ideas with which the student is familiar and the derivation of such formulas from rules expressed in words.

The dependence of one variable upon another. Numerous illustrations and problems involving the linear function $y = mx + b$. Illustrations and problems involving the quadratic function $y = kx^2$.

Graphs and graphic representations in general; their construction and interpretation, including the representation of statistical data and the use of the graph to exhibit dependence.

Positive and negative numbers; their meaning and use.

Linear equations in one unknown quantity; their use in solving problems.

Sets of linear equations involving two unknown quantities; their use in solving problems.

Ratio, as a case of simple fractions; proportion without the theorems on alternation, etc.; and simple cases of variation.

The essentials of algebraic technique. This should include —

(*a*) The four fundamental operations.

(*b*) Factoring of the following types: Common factors of the terms of a polynomial; the difference of two squares; trinomials of the second degree (including the square of a binomial) that can be easily factored by trial.

(*c*) Fractions, including complex fractions of a simple type.

(*d*) Exponents and radicals. The laws for positive integral exponents; the meaning and use of fractional exponents, but not the formal theory. The consideration of radicals may be confined to the simplification of

expressions of the form $\sqrt{a^2b}$ and $\sqrt{a/b}$ and to the evaluation of simple expressions involving the radical sign. A process for extracting the square root of a number should be included but not the process for extracting the square root of a polynomial.

Numerical trigonometry. The use of the sine, cosine, and tangent in solving right triangles. The use of three or four place tables of natural functions.

Major requirement. In addition to the minor requirement as specified above, the following should be included:

Illustrations and problems involving the quadratic function $y = ax^2 + bx + c$.

Quadratic equations in one unknown; their use in solving problems.

Exponents and radicals. Zero and negative exponents, and more extended treatment of fractional exponents. Rationalizing denominators. Solution of simple types of radical equations.

The use of logarithmic tables in computation without the formal theory.

Elementary statistics, including a knowledge of the fundamental concepts and simple frequency distributions, with graphic representations of various kinds.

The binomial theorem for positive integral exponents less than 8; with such applications as compound interest.

The formula for the nth term, and the sum of n terms, of arithmetic and geometric progressions, with applications.

Simultaneous linear equations in three unknown quantities and simple cases of simultaneous equations involving one or two quadratic equations; their use in solving problems.

Drill in algebraic manipulation should be limited, particularly in the minor requirement, by the purpose of securing a thorough understanding of important principles and facility in carrying out those processes which are fundamental and of frequent occurrence either in common life or in the subsequent courses that a substantial proportion of the pupils will study. Skill in manipulation must be conceived of throughout as a means to an end, not as an end in itself. Within these limits, skill and accuracy in algebraic technique are of prime importance, and drill in this subject should be extended far enough to enable students to carry out the fundamentally essential processes accurately and with reasonable speed.

The consideration of literal equations, when they serve a significant purpose, such as the transformation of formulas, the derivation of a general solution (as of the quadratic equation), or the proof of a theorem, is important. As a means for drill in algebraic technique they should be used sparingly.

The solution of problems should offer opportunity throughout the course for considerable arithmetical and computational work. The conception of algebra as an extension of arithmetic should be made significant both in numerical applications and in elucidating algebraic principles. Emphasis should be placed upon the use of common sense and judgment in computing from approximate data, especially with regard to the number of figures retained, and on the necessity for checking the results. The use of tables to facilitate computation (such as tables of squares and square roots, of interest, and of trigonometric functions) should be encouraged.

PLANE GEOMETRY

The usual theorems and constructions of good textbooks, including the general properties of plane rectilinear figures; the circle and the measurement of angles; similar polygons; areas; regular polygons and the measurement of the circle. The solution of numerous original exercises, including locus problems. Applications to the mensuration of lines and plane surfaces.

The scope of the required work in plane geometry is indicated by the List of Fundamental Propositions and Constructions, which is given in the next chapter. This list indicates in Section I the type of proposition which, in the opinion of the committee, may be assumed without proof or given informal treatment. Section II contains 52 propositions and 19 constructions which are regarded as so fundamental that they should constitute the common minimum of all standard courses in plane geometry. Section III gives a list of subsidiary theorems which suggests the type of additional propositions that should be included in such courses.

College entrance examinations. College entrance examinations exert in many schools, and especially throughout the eastern section of the country, an influence on secondary school teaching which is very far-reaching. It is, therefore, well within the province of the National Committee to inquire whether the prevailing type of examination in mathematics serves the best interests of mathematical education and of college preparation.

The reason for the almost controlling influence of entrance examinations in the schools referred to is readily recognized. Schools sending students to such colleges for men as Harvard, Yale, and Princeton, to the larger

colleges for women, or to any institution where examinations form the only or prevailing mode of admission, inevitably direct their instruction toward the entrance examination. This remains true even if only a small percentage of the class intends to take these examinations, the point being that the success of a teacher is often measured by the success of his or her students in these examinations.

In the judgment of the committee, the prevailing type of entrance examination in algebra is primarily a test of the candidate's skill in formal manipulation, and not an adequate test of his understanding or of his ability to apply the principles of the subject. Moreover, it is quite generally felt that the difficulty and complexity of the formal manipulative questions, which have appeared on recent papers set by colleges and by such agencies as the College Entrance Examination Board, has often been excessive. As a result, teachers preparing pupils for these examinations have inevitably been led to devote an excessive amount of time to drill in algebraic technique, without insuring an adequate understanding of the principles involved. Far from providing the desired facility, this practice has tended to impair it. For "practical skill, modes of effective technique, can be intelligently, non-mechanically used only when intelligence has played a part in their acquisition." (Dewey: *How We Think*, p. 52.)

Moreover, it must be noted that authors and publishers of textbooks are under strong pressure to make their content and distribution of emphasis conform to the prevailing type of entrance examination. Teachers in turn are too often unable to rise above the textbook. An

improvement in the examinations in this respect will cause a corresponding improvement in textbooks and in teaching.

On the other hand, the makers of entrance examinations in algebra cannot be held solely responsible for the condition described. Theirs is a most difficult problem. Not only can they reply that as long as algebra is taught as it is, examinations must be largely on technique,[6] but they can also claim with considerable force that technical facility is the only phase of algebra that can be fairly tested by an examination; that a candidate can rarely do himself justice amid unfamiliar surroundings and subject to a time limit on questions involving real thinking in applying principles to concrete situations; and that we must face here a real limitation on the power of an examination to test attainment. Many, and perhaps most, teachers will agree with this claim. Past experience is on their side; no generally accepted and effective "power test" in mathematics has as yet been devised and, if devised, it might not be suitable for use under conditions prevailing during an entrance examination.

But if it is true that the power of an examination is thus inevitably limited, the wisdom and fairness of using it as the sole means of admission to college is surely open to grave doubt. That many unqualified candidates are admitted under this system is not open to question. Is it not probable that many qualified candidates are at the same time excluded? If the entrance examination is a fair test of manipulative skill only, should not the colleges

[6] The vicious circle is now complete. Algebra is taught mechanically because of the character of the entrance examination; the examination, in order to be fair, must conform to the character of the teaching.

use additional means for learning something about the candidate's other abilities and qualifications?

Some teachers believe that an effective "power test" in mathematics is possible. Efforts to devise such a test should receive every encouragement.

In the meantime, certain desirable modifications of the prevailing type of entrance examination are possible. The College Entrance Examination Board recently appointed a committee to consider this question and a conference [7] on this subject was held by representatives of the College Entrance Examination Board, members of the National Committee, and others. The following recommendations are taken from the report of the committee just referred to:

> Fully one third of the questions should be based on topics of such fundamental importance that they will have been thoroughly taught, carefully reviewed, and deeply impressed by effective drill. . . . They should be of such a degree of difficulty that any pupil of regular attendance, faithful application, and even moderate ability may be expected to answer them satisfactorily.
>
> There should be both simple and difficult questions testing the candidate's ability to apply the principles of the subject. The early ones of the easy questions should be really easy for the candidate of good average ability who can do a little thinking under the stress of an examination; but even these questions should have genuine scientific content.
>
> There should be a substantial question which will put the

[7] At this conference the following vote was unanimously passed: "Voted, that the results of examinations (of the College Entrance Examination Board), be reported by letters A, B, C, D, E, and that the definition of the groups represented by these letters should be determined in each year by the distribution of ability in a standard group of papers representing widely both public and private schools."

best candidates on their mettle, but which is not beyond the reach of a fair proportion of the really good candidates. This question should test the normal workings of a well-trained mind. It should be capable of being thought out in the limited time of the examination. It should be a test of the candidate's grasp and insight — not a catch question or a question of unfamiliar character making extraordinary demands on the critical powers of the candidate or one the solution of which depends on an inspiration. Above all, this question should lie near to the heart of the subject as all well-prepared candidates understand the subject.

As a rule, a question should consist of a single part and be framed to test one thing — not pieced together out of several unrelated and perhaps unequally important parts.

Each question should be a substantial test on the topic or topics which it represents. It is, however, in the nature of the case impossible that all questions be of equal value.

Care should be used that the examination be not too long. ... The examiner should be content to ask questions on the important topics, so chosen that their answers will be fair to the candidate and instructive to the readers; and beyond this merely to sample the candidate's knowledge of the minor topics.

The National Committee suggests the following additional principles: The examination as a whole should, as far as practicable, reflect the principle that algebraic technique is a means to an end, and not an end in itself.

Questions that require of the candidate skill in algebraic manipulation beyond the needs of actual application should be used very sparingly.

An effort should be made to devise questions which will fairly test the candidate's understanding of principles and his ability to apply them, while involving a minimum of manipulative complexity.

The examinations in geometry should be definitely constructed to test the candidate's ability to draw valid conclusions rather than his ability to memorize an argument.

A chapter on mathematical terms and symbols is included in this report.[8] It is hoped that examining bodies will be guided by the recommendations there made relative to the use of terms and symbols in elementary mathematics.

The College Entrance Examination Board, early in 1921, appointed a commission to recommend such revisions as might seem necessary in the definitions of the requirements in the various subjects of elementary mathematics. The recommendations contained in the present chapter have been laid before this commission. It is hoped that the commission's report, when it is finally made effective by action of the College Entrance Examination Board and the various colleges concerned, will give impetus to the reorganization of the teaching of elementary mathematics along the lines recommended in the report of the National Committee.[9]

[8] See Chapter VIII.

[[9] This hope has been fully realized. See the present C.E.E.B. requirements on p. 140.]

VI

LIST OF PROPOSITIONS IN PLANE AND SOLID GEOMETRY

General basis of the selection of material. The subcommittee appointed to prepare a list of basal propositions made a careful study of a number of widely used textbooks on geometry. The bases of selection of the propositions were two: (1) The extent to which the propositions and corollaries were used in subsequent proofs of important propositions and exercises; (2) the value of propositions in completing important pieces of theory. Although the list of theorems and problems is substantially the same in nearly all textbooks in general use in this country, the wording, the sequence, and the methods of proof vary to such an extent as to render difficult a definite statement as to the number of times a proposition is used in the several books examined. A tentative table showed, however, less variation than might have been anticipated.

Classification of propositions. The classification of propositions is not the same in plane geometry as in solid geometry. This is partly due to the fact that it is generally felt that the student should limit his construction work to figures in a plane and in which the compasses and straight edge are sufficient. The propositions have been divided as follows:

Plane geometry: I. Assumptions and theorems for informal treatment; II. Fundamental theorems and constructions: A. Theorems, B. Constructions; III. Subsidiary theorems.

Solid geometry: I. Fundamental theorems; II. Fundamental propositions in mensuration; III. Subsidiary theorems; IV. Subsidiary propositions in mensuration.[1]

Plane Geometry

I. Assumptions and theorems for informal treatment. This list contains propositions which may be assumed without proof (postulates) and theorems which it is permissible to treat informally. Some of these propositions will appear as definitions in certain methods of treatment. Moreover, teachers should feel free to require formal proofs in certain cases, if they desire to do so. The precise wording given is not essential, nor is the order in which the propositions are here listed. The list should be taken as representative of the type of propositions which may be assumed, or treated informally, rather than as exhaustive.

1. Through two distinct points it is possible to draw one straight line, and only one.
2. A line segment may be produced to any desired length.
3. The shortest path between two points is the line segment joining them.
4. One and only one perpendicular can be drawn through a given point to a given straight line. [6*, not more than one perpendicular.]
5. The shortest distance from a point to a line is the perpendicular distance from the point to the line.
6. From a given center and with a given radius one and only one circle can be described in a plane.

[[1] In the lists below the numbers in [ ] give the numbers of the corresponding propositions in the C.E.E.B. lists. The letters "cd" after a proposition indicates that it is included in the C.E.E.B. requirement for mathematics, cd, Plane and Solid Geometry, Minor Requirement. See pp. 162, 168, for meaning of asterisks following some of the numbers.]

7. A straight line intersects a circle in at most two points.

8. Any figure may be moved from one place to another without changing its shape or size.

9. All right angles are equal.

10. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.

11. Equal angles have equal complements and equal supplements.

12. Vertical angles are equal.

13. Two lines perpendicular to the same line are parallel.

[7]

14. Through a given point not on a given straight line, one straight line, and only one, can be drawn parallel to the given line.

15. Two lines parallel to the same line are parallel to each other.

16. The area of a rectangle is equal to its base times its altitude.

II. Fundamental theorems and constructions. It is recommended that theorems and constructions (other than originals) to be proved on college entrance examinations be chosen from the following list. Originals and other exercises should be capable of solution by direct reference to one or more of these propositions and constructions. It should be obvious that any course in geometry that is capable of giving adequate training must include considerable additional material. The order here given is not intended to signify anything as to the order of presentation. It should be clearly understood that certain of the statements contain two or more theorems, and that the precise wording is not essential. The committee favors entire freedom in statement and sequence.

LIST OF PROPOSITIONS IN GEOMETRY

A. THEOREMS

1. Two triangles are congruent if [2] (*a*) two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other; (*b*) two angles and a side of one are equal, respectively, to two angles and the corresponding side of the other; (*c*) the three sides of one are equal respectively, to the three sides of the other. [1*, 2*, 5*, cd*]

2. Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other. [10*, cd*]

3. If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely.[3] [3*, 4*, cd*]

4. The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line joining them. [30*, cd*]

5. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines. [31*, cd*]

6. When a transversal cuts two parallel lines, the alternate interior angles are equal; and conversely. [11*, cd*, 14, cd]

7. The sum of the angles of a triangle is two right angles. [23*, cd*]

8. A parallelogram is divided into congruent triangles by either diagonal.

9. Any (convex) quadrilateral is a parallelogram (*a*) if the

[2] Teachers should feel free to separate this theorem into three distinct theorems and to use other phraseology for any such proposition. For example, in 1, "Two triangles are equal if . . . ," "a triangle is determined by . . . ," etc. Similarly in 2, the statement might read: "Two right triangles are congruent if, besides the right angles, any two parts (not both angles) in the one are equal to corresponding parts of the other."

[3] It should be understood that the converse of a theorem need not be treated in connection with the theorem itself, it being sometimes better to treat it later. Furthermore, a converse may occasionally be accepted as true in an elementary course, if the necessity for proof is made clear. The proof may then be given later.

31.[3] The area of a circle is equal to πr^2. (Informal proof only.) [88, cd]

The treatment of the mensuration of the circle should be based upon related theorems concerning regular polygons, but it should be informal as to the limiting processes involved. The aim should be an understanding of the concepts involved, so far as the capacity of the pupil permits.

B. CONSTRUCTIONS

1. Bisect a line segment [1, cd] and draw the perpendicular bisector.
2. Bisect an angle. [2, cd]
3. Construct a perpendicular to a given line through a given point. [3, 4, cd]
4. Construct an angle equal to a given angle. [5, cd]
5. Through a given point draw a straight line parallel to a given straight line. [6, cd]
6. Construct a triangle, given (*a*) the three sides; (*b*) two sides and the included angle; (*c*) two angles and the included side. [8, 9, 10, cd]
7. Divide a line segment into parts proportional to given segments. [7, cd (equal parts), 15, cd]
8. Given an arc of a circle, find its center.
9. Circumscribe a circle about a triangle. [11, cd]
10. Inscribe a circle in a triangle. [12, cd]
11. Construct a tangent to a circle through a given point. [13, 14, cd]
12. Construct the fourth proportional to three given line segments. [16, cd]
13. Construct the mean proportional between two given line segments. [17, cd]
14. Construct a triangle (polygon) similar to a given triangle (polygon). [18]

[3] The total number of theorems given in this list when separated, as will probably be found advantageous in teaching, including the converses indicated, is 52.

15. Construct a triangle equal to a given polygon.
16. Inscribe a square in a circle. [19, cd]
17. Inscribe a regular hexagon in a circle. [20, cd]

III. Subsidiary list of propositions. The following list of propositions is intended to suggest some of the additional material referred to in the introductory paragraph of Section II. It is not intended, however, to be exhaustive; indeed, the committee feels that teachers should be allowed considerable freedom in the selection of such additional material, theorems, corollaries, originals, exercises, etc., in the hope that opportunity will thus be afforded for constructive work in the development of courses in geometry.

1. When two lines are cut by a transversal, if the corresponding angles are equal [15, cd] or if the interior angles on the same side of the transversal are supplementary [16], the lines are parallel.
2. When a transversal cuts two parallel lines, the corresponding angles are equal [12, cd], and the interior angles on the same side of the transversal are supplementary. [13]
3. A line perpendicular to one of two parallels is perpendicular to the other also. [8, cd]
4. If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary. [17, cd]
5. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles. [24, cd]
6. The sum of the angles of a convex polygon of n sides is $2(n - 2)$ right angles. [25]
7. In any parallelogram (*a*) the opposite sides are equal; (*b*) the opposite angles are equal; (*c*) the diagonals bisect each other. [18, 19, cd]
8. Any (convex) quadrilateral is a parallelogram, if (*a*) the opposite angles are equal; (*b*) the diagonals bisect each other.

9. The medians of a triangle intersect in a point which is two thirds of the distance from a vertex to the mid-point of the opposite side. [35]

10. The altitudes of a triangle meet in a point. [34]

11. The perpendicular bisectors of the sides of a triangle meet in a point. [32]

12. The bisectors of the angles of a triangle meet in a point. [33]

13. The tangents to a circle from an external point are equal. [45, cd, part]

14.[4] (*a*) If two sides of a triangle are unequal, the greater side has the greater angle opposite it, and conversely. [26, 27, cd]

(*b*) If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second, and conversely. [28, 29]

(*c*) If two chords are unequal, the greater is at the less distance from the center, and conversely. [55, 56]

(*d*) The greater of two minor arcs has the greater chord, and conversely. [53, 54]

15. An angle inscribed in a semicircle is a right angle. [49, cd]

16. Parallel lines cutting a circle intercept equal arcs on the circle. [46, cd]

17. An angle formed by a tangent of a circle and a chord drawn through the point of contact is measured by half the intercepted arc. [50, cd]

18. An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs. [51]

[4] Such inequality theorems as these are of importance in developing the notion of dependence or functionality in geometry. The fact that they are placed in the "Subsidiary list of propositions" should not imply that they are considered of less educational value than those in List II. They are placed here because they are not "fundamental" in the same sense that the theorems of List II are fundamental.

19. An angle formed by two secants or by two tangents to a circle is measured by half the difference between the intercepted arcs. [52, part]

20. If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment. [68]

21. Parallelograms or triangles of equal bases and equal altitudes are equal. [74, cd]

22. The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems. [83] [5]

Solid Geometry

In the following list the precise wording and the sequence are not considered:

I. FUNDAMENTAL THEOREMS

1. If two planes meet, they intersect in a straight line. [1, cd]

2. If a line is perpendicular to each of two intersecting lines at their point of intersection it is perpendicular to the plane of the two lines. [2*, cd*]

3. Every perpendicular to a given line at a given point lies in a plane perpendicular to the given line at the given point. [3, cd]

4. Through a given point (internal or external) there can pass one and only one line perpendicular to a plane. [6*, 7*, cd*]

5. Two lines perpendicular to the same plane are parallel. [10*, cd*]

6. If two lines are parallel, every plane containing one of the lines and only one is parallel to the other. [13, cd]

7. Two planes perpendicular to the same line are parallel. [16, cd]

[[5] For additional theorems in Plane Geometry contained in the C.E.E.B List see p. 169.]

8. If two parallel planes are cut by a third plane, the lines of intersection are parallel. [17, cd]

9. If two angles not in the same plane have their sides respectively parallel in the same sense, they are equal and their planes are parallel. [21*, cd*]

10. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other. [23*, cd*]

11. If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane. [26*, cd*]

12. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane. [27*, cd]

13. The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons. [37*]

14. An oblique prism is equal to a right prism whose base is equal to a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism. [40*]

15. The opposite faces of a parallelepiped are congruent. [42]

16. The plane passed through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equal triangular prisms. [43*]

17. If a pyramid (or a cone) is cut by a plane parallel to the base:

(*a*) The lateral edges (or elements) and the altitude are divided proportionally; [47*, cd* (pyramid)]

(*b*) The section is a figure similar to the base; [47*, cd* (pyramid), 56* (cone)]

(*c*) The area of the section is to the area of the base as the square of the distance from the vertex is to the square of the altitude of the pyramid (or cone).

18. Two triangular pyramids having equal bases and equal altitudes are equal. [50]

19. All points on a circle of a sphere are equidistant from either pole of the circle. [67]

20. On any sphere a point which is at a quadrant's distance from each of two other points not the extremities of a diameter is a pole of the great circle passing through these two points. [68*]

21. If a plane is perpendicular to a radius at its extremity on a sphere, it is tangent to the sphere. [69; and conversely, 70]

22. A sphere can be inscribed in [71] or circumscribed about [72*] any tetrahedron.

23. If one spherical triangle is the polar of another, then reciprocally the second is the polar triangle of the first. [76*]

24. In two polar triangles each angle of either is the supplement of the opposite side of the other. [77*]

25. Two symmetric spherical triangles are equal. [79]

II. FUNDAMENTAL PROPOSITIONS IN MENSURATION

26. The lateral area of a prism [41*] or a circular cylinder [54, cd] is equal to the product of a lateral edge or element, respectively, by the perimeter (circumference) of a right section.

27. The volume of a prism [46, cd; triangular, 45*] (including any parallelepiped [44, cd]) or of a circular cylinder [55, cd] is equal to the product of its base by its altitude.

28. The lateral area of a regular pyramid [48*] or a right circular cone [57, cd] is equal to half the product of its slant height by the perimeter (circumference) of its base.

29. The volume of a pyramid [52, cd; triangular, 51*] or a cone [59, cd] is equal to one third the product of its base by its altitude.

30. The area of a sphere. [91, cd]

31. The area of a spherical polygon.

32. The volume of a sphere. [92, cd]

III. SUBSIDIARY THEOREMS

33. If from an external point a perpendicular and obliques are drawn to a plane, (*a*) the perpendicular is shorter than any oblique [8, cd]; (*b*) obliques meeting the plane at equal distances from the foot of the perpendicular are equal [9]; (*c*) of two obliques meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer. [9]

34. If two lines are cut by three parallel planes, their corresponding segments are proportional. [22]

35. Between two lines not in the same plane there is one common perpendicular, and only one. [29, cd]

36. The bases of a cylinder are congruent. [53]

37. If a plane intersects a sphere, the line of intersection is a circle. [64*]

38. The volume of two tetrahedrons that have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.

39. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.

40. Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.

41. The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.

42. The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges. [61]

43. If three face angles of one trihedral angle are equal, respectively, to the three face angles of another the trihedral angles are either congruent or symmetric. [36]

44. Two spherical triangles on the same sphere are either congruent or symmetric if (*a*) two sides and the included angle of one are equal to the corresponding parts of the other; (*b*) two angles and the included side of one are equal to the corresponding parts of the other; (*c*) they are mutually equilateral; (*d*) they are mutually equiangular. [80, 81, 82, 83]

45. The sum of any two face angles of a trihedral angle is greater than the third face angle. [33]

46. The sum of the face angles of any convex polyhedral angle is less than four right angles. [34]

47. Each side of a spherical triangle is less than the sum of the other two sides. [74]

48. The sum of the sides of a spherical polygon is less than 360°. [75]

49. The sum of the angles of a spherical triangle is greater than 180° and less than 540°. [78*]

50. There cannot be more than five regular convex polyhedrons. [35]

51. The locus of points equidistant (*a*) from two given points; (*b*) from two given planes which intersect. [30, 32]

IV. SUBSIDIARY PROPOSITIONS IN MENSURATION

52. The volume of a frustum of (*a*) a pyramid or (*b*) a cone.

53. The lateral area of a frustum of (*a*) a pyramid [regular, 49] or (*b*) a cone of revolution. [58] [6]

54. The volume of a prismoid (without formal proof).

[6] For additional theorems in Solid Geometry contained in the C.E.E.B. List see p. 169.

VII

THE FUNCTION CONCEPT IN SECONDARY SCHOOL MATHEMATICS [1]

In Chapter II, and incidentally in later chapters, considerable emphasis has been placed on the function concept or, better, on the idea of relationship between variable quantities as one of the general ideas that should dominate instruction in elementary mathematics. Since this recommendation is peculiarly open to misunderstanding on the part of teachers, it seems desirable to devote a separate chapter to a rather detailed discussion of what the recommendation means and implies.

It will be seen in what follows that there is no disposition to advocate the teaching of any sort of function *theory*. A prime danger of misconception that should be removed at the very outset is that teachers may think it is the notation and the definitions of such a theory that are to be taught. Nothing could be further from the intention of the committee. Indeed, it seems entirely safe to say that the word "function" had best not be used at all in the early courses.

What is desired is that the idea of relationship or dependence between variable quantities be imparted to the

[1] The first draft of this chapter was prepared for the National Committee by E. R. Hedrick, of the University of Missouri [at present, of the University of California, Southern Branch]. It was discussed at the meeting of the Committee, September 2–4, 1920; revised by the author, and again discussed December 29, 30, 1920, and is now issued as part of the committee's report.

pupil by the examination of numerous concrete instances of such relationship. He must be shown the workings of relationships in a large number of cases before the abstract idea of relationship will have any meaning for him. Furthermore, the pupil should be led to form the habit of thinking about the connections that exist between related quantities, not merely because such a habit forms the best foundation for a real appreciation of the theory that may follow later, but chiefly because this habit will enable him to think more clearly about the quantities with which he will have to deal in real life, whether or not he takes any further work in mathematics.

Indeed, the reason for insisting so strongly upon attention to the idea of relationships between quantities is that such relationships do occur in real life in connection with practically all of the quantities with which we are called upon to deal in practice. Whereas there can be little doubt about the small value to the student who does not go on to higher studies of some of the manipulative processes criticized by the National Committee, there can be no doubt at all of the value to all persons of any increase in their ability to see and to foresee the manner in which related quantities affect each other.

To attain what has been suggested, the teacher should have in mind constantly not any definition to be recited by the pupil, not any automatic response to a given cue, not any memory exercise at all, but rather a determination not to pass any instance in which one quantity is related to another, or in which one quantity is determined by one or more others, without calling attention to the fact, and trying to have the student "see how it works."

These instances occur in literally thousands of cases in both algebra and geometry. It is the purpose of this chapter to outline in some detail a few typical instances of this character.

I. Relationships in Algebra

In algebra the instance of the function idea which usually occurs to one first is in connection with the study of graphs. While this is natural enough, and while it is true that the graph is fundamentally functional in character, the supposition that it furnishes the first opportunity for observing functional relations between quantities betrays a misconception that ought to be corrected.

1. Use of letters for numbers. The very first illustrations given in algebra to show the use of letters in the place of numbers are essentially functional in character. Thus, such relations as $I = prt$ and $A = \pi r^2$, as well as others that are frequently used, are statements of general relationships. These should be used to accustom the student not only to the use of letters in the place of numbers and to the solution of simple numerical problems, but also to the idea, for example, that changes in r affect the value of A. Such questions as the following should be considered: If r is doubled, what will happen to A? If p is doubled, what will happen to I? Appreciation of the meaning of such relationships will tend to clarify the entire subject under consideration. Without such an appreciation, it may be doubted whether the student has any real grasp of the matter.

2. Equations. Every simple problem leading to an equation in the first part of algebra would be better

understood for just such a discussion as that mentioned above. Thus, if two dozen eggs are weighed in a basket which weighs 2 pounds, and if the total weight is found to be 5 pounds, what is the average weight of an egg? If x is the weight in ounces of one egg, the total weight with the 2-pound basket would be $24x + 32$ ounces. If the student will first try the effect of an average weight of 1 ounce, of $1\frac{1}{2}$ ounces, 2 ounces, $2\frac{1}{2}$ ounces, the meaning of the problem will stand out clearly. In every such problem, preliminary trials really amount to a discussion of the properties of a linear function.

3. Formulas of pure science and of practical affairs. The study of formulas as such, aside from their numerical evaluation, is becoming of more and more importance. The actual uses of algebra are not to be found solely nor even principally in the solution of numerical problems for numerical answers. In such formulas as those for falling bodies, levers, etc., the manner in which changes in one quantity cause (or correspond to) changes in another, are of prime importance, and their discussion need cause no difficulty whatever. The formulas under discussion here include those formulas of pure science and of practical affairs which are being introduced more and more into our texts on algebra. Whenever such a formula is encountered, the teacher should be sure that the students have some comprehension of the effects of changes in one of the quantities upon the other quantity or quantities in the formula.

As a specific instance of such scientific formulas, consider, for example, the force F, in pounds, with which a weight W, in pounds, pulls outward on a string (centrifugal force) if the weight is revolved rapidly at a speed v,

in feet per second, at the end of a string of length r feet. This force is given by the formula

$$F = \frac{W\,v^2}{32\,r}$$

When such a formula is used the teacher should not be contented with the mere insertion of numerical values for W, v, and r to obtain a numerical value for F.

The advantage obtained from the study of such a formula lies quite as much in the recognition of the behavior of the force when one of the other quantities varies. Thus the student should be able to answer intelligently such questions as the following: If the weight is assumed to be twice as heavy, what is the effect upon the force? If the speed is taken twice as great, what is the effect upon the force? If the radius becomes twice as large, what is the effect upon the force? If the speed is doubled, what change in the weight would result in the same force? Will an increase in the speed cause an increase or decrease in the force? Will an increase in the radius r cause an increase or a decrease in the force?

As another instance (of a more advanced character) consider the formula for the amount of a sum of money P, at compound interest at r per cent, at the end of n years. This amount may be denoted by A_n. Then we shall have $A_n = P\,(1 + r)^n$. Will doubling P result in doubling A_n? Will doubling n result in doubling A_n? Since the compound interest that has accumulated is equal to the difference between P and A_n, will the doubling of r double the interest? Compare the correct answers to these questions with the answers to the similar questions in the case of simple interest, in which the formula reads

$A_n = P + Prn$ and in which the accumulated interest is simply Prn.

The difference between such a study of the effect produced upon one quantity by changes in another and the mere substitution of numerical values will be apparent from these examples.

4. **Formulas of pure algebra.** Formulas of pure algebra, such as that for $(x + h)^2$, will be better understood and appreciated if accompanied by a discussion of the manner in which changes in h cause changes in the total result. This can be accomplished by discussing such concrete realities as the error made in computing the area of a square field or of a square room when an error has been made in measuring the side of the square. If x is the true length of the side, and if the student assumes various possible values for the error h made in measuring x, he will have a situation that involves some comprehension of the functional workings of the formula mentioned. The same formula relates to such problems as the change from one entry to the next entry in a table of squares.

A similar situation, and a very important one, occurs with the pure algebraic formula for $(x + a)(y + b)$. This formula may be said to govern the question of the keeping of significant figures in finding the product xy. For, if a and b represent the uncertainty in x and y, respectively, the uncertainty in the product is given by this formula. The student has much to learn on this score, for the retention of meaningless figures in a product is one of the commonest mistakes of both student and teacher in computational work.

Such formulas occur throughout algebra, and each of them will be illuminated by such a discussion. The for-

mulas for arithmetic and geometric progression, for example, should be studied from a functional standpoint.

5. **Tables.** The uses of the functional idea in connection with numerical computation have already been mentioned in connection with the formula for a product. Work which appears on the surface to be wholly numerical may be of a distinctly functional character. Thus, any table, e.g., a table of squares, corresponds to or is constructed from a functional relation, e.g., for a table of squares, the relation $y = x^2$. The differences in such a table are the differences caused by changes in the values of the independent variable. Thus, the differences in a table of squares are precisely the differences between x^2 and $(x + h)^2$ for various values of x.

6. **Graphs.** The functional character of graphic representations was mentioned at the beginning of this section. Every graph is obviously a representation of a functional relationship between two or more quantities. What is needed is only to draw attention to this fact and to study each graph from this standpoint. In addition to this, however, it is desirable to point out that functional relations may be studied directly by means of graphs without the intervention of any algebraic formula. Thus, such a graph as a population curve, or a curve representing wind pressure, obviously represents a relationship between two quantities, but there is no known formula in either case. The idea that the three concepts, tables, graphs, algebraic formulas, are all representations of the same kind of connection between quantities, and that we may start in some instances with any of the three, is a most valuable addition to the student's mental equipment, and to his

control over the quantities with which he will deal in his daily life.

II. Relationships in Geometry

Thus far the instances mentioned have been largely algebraic, though certain mensuration formulas of geometry have been mentioned. While the mensuration formulas may occur to one first as an illustration of functional concepts in geometry, they are by no means the earliest relationships that occur in that study.

1. **Congruence.** Among the earliest theorems are those on the congruence of triangles. In any such theorem, the parts necessary to establish congruence evidently determine completely the size of each other part. Thus, two sides and the included angle of a triangle evidently determine the length of the third side. If the student clearly grasps this fact, the meaning of this case of congruence will be more vivid to him, and he will be prepared for its important applications in surveying and in trigonometry. Even if he never studies these subjects, he will nevertheless be able to use his understanding of the situation in any practical cases in which the angle between two fixed rods or beams is to be fixed or is to be determined, in a practical situation such as house building. Other congruence theorems throughout geometry may well be treated in a similar manner.

2. **Inequalities.** In the theorems regarding inequalities, the functional quality is even more pronounced. Thus, if two triangles have two sides of one equal respectively to two sides of the other, but if the included angle between these sides in the one triangle is greater than the corresponding angle in the other, then the third

sides of the triangles are unequal in the same sense. This theorem shows that as one angle grows, the side opposite it grows, if the other sides remain unchanged. A full realization of the fact here mentioned would involve a real grasp of the functional relation between the angle and the side opposite it. Thus, if the angle is doubled, will the side opposite it be doubled? Such questions arise in connection with all theorems on inequalities.

3. **Variations in figures.** A great assistance to the imagination is gained in certain figures by imagining variations of the figure through all intermediate stages from one case to another. Thus, the angle between two lines that cut a circle is measured by a proper combination of the two arcs cut out of the circle by the two lines. As the vertex of the angle passes from the center of the circle to the circumference and thence to the outside of the circle, the rule changes, but these changes may be borne in mind, and the entire scheme may be grasped, by imagining a continuous change from the one position to the other, following all the time the changes in the intercepted arcs. The angle between a secant and a tangent is measured in a manner that can best be grasped by another such continuous motion, watching the changes in the measuring arcs as the motion occurs. Such observations are essentially functional in character, for they consist in careful observations of the relationships between the angle to be measured and the arcs that measure it.

4. **Motion.** The preceding discussion of variable figures leads naturally to a discussion of actual motion. As figures move, either in whole or in part, the relationships between the quantities involved may change. To note these changes is to study the functional relationships be-

tween the parts of the figures. Without the functional idea, geometry would be wholly *static*. The study of fixed figures should not be the sole purpose of a course in geometry, for the uses of geometry are not wholly on static figures. Indeed, in all machinery, the geometric figures formed are in continual motion, and the shapes of the figures formed by the moving parts change. The study of motion and of moving forms, the *dynamic* aspects of geometry, should be given at least some consideration. Whenever this is done, the functional relations between the parts become of prime importance. Thus a linkage of the form of a parallelogram can be made more nearly rectangular by making the diagonals more nearly equal, and the linkage becomes a rectangle if the diagonals are made exactly equal. This principle is used in practice in making a rectangular framework precisely true.

5. **Proportionality theorems.** All theorems which assert that certain quantities are in proportion to certain others, are obviously functional in character. Thus even the simplest theorems on rectangles assert that the area of a rectangle is directly proportional to its height, if the base is fixed. When more serious theorems are reached, such as the theorems on similar triangles, the functional ideas involved are worthy of considerable attention. That this is eminently true will be realized by all to whom trigonometry is familiar, for the trigonometric functions are nothing but the ratios of the sides of right triangles. But even in the field of elementary geometry a clear understanding of the relation between the areas (and volumes) of similar figures and the corresponding linear dimensions is of prime importance.

III. Relationships in Trigonometry

The existence of functional relationships in trigonometry is evidenced by the common use of the words "trigonometric functions" to describe the trigonometric ratios. Thus the sine of an angle is a definite ratio whose value depends upon and is determined by the size of the angle to which it refers. The student should be made conscious of this relationship and he should be asked such questions as the following: Does the sine of an angle increase or decrease as the angle changes from zero to 90°? If the angle is doubled, does the sine of the angle double? If not, is the sine of double the angle more or less than twice the sine of the original angle? How does the value of the sine behave as the angles increase from 90° to 180°? From 180° to 270°? From 270° to 360°? Similar questions may be asked for the cosine and for the tangent of an angle.

Such questions may be reinforced by the use of figures that illustrate the points in question. Thus an angle twice a given angle should be drawn, and its sine should be estimated from the figure. A central angle and an inscribed angle on the same arc may be drawn in any circle. If they have one side in common, the relations between their sines will be more apparent. Finally, the relationships that exist may be made vivid by actual comparison of the numerical values found from the trigonometric tables.

Not only in these first functional definitions, however, but in a variety of geometric figures throughout trigonometry do functional relations appear. Thus the law of cosines states a definite relationship between the three sides of a triangle and any one of the angles. How will

the angle be affected by an increase or decrease of the side opposite it, if the other two sides remain fixed? How will the angle be affected by an increase or a decrease of one of the adjacent sides, if the other two sides remain fixed? Are these statements still true, if the angle in question is obtuse?

As another example, the height of a tree, or the height of a building, may be determined by measuring the two angles of elevation from two points on the level plain in a straight line with its base. A formula for the height (h) in terms of these two angles (A, B) and the distance (d) between the points of observation, may be easily written down [$h = d \sin A \sin B / \sin (A - B)$]. Then the effect upon the height of changes in one of these angles may be discussed.

In a similar manner, every formula that is given or derived in a course on trigonometry may be discussed with profit from the functional standpoint.

IV. Conclusion

In conclusion, mention should be made of the great rôle which the idea of functions plays in the life of the world about us. Even when no calculation is to be carried out, the problems of real life frequently involve the ability to think correctly about the nature of the relationships which exist between related quantities. Specific mention has been made already of this type of problem in connection with interest on money. In everyday affairs, such as the filling-out of formulas for fertilizers or for feeds or for spraying mixtures on the farm, the similar filling-out of recipes for cooking (on different scales from that of the book of recipes), or the proper balancing of the

ration in the preparation of food, many persons are at a loss on account of their lack of training in thinking about the relations between quantities. Another such instance of very common occurrence in real life is in insurance. Very few men or women attempt intelligently to understand the meaning and the fairness of premiums on life insurance and on other forms of insurance, chiefly because they cannot readily grasp the relations of interest and of chance that are involved. These relations are not particularly complicated, and they do not involve any great amount of calculation for the comprehension of the meaning and of the fairness of the rates. Mechanics, farmers, merchants, housewives, as well as scientists and engineers have to do constantly with quantities of things, and the quantities with which they deal are related to other quantities in ways that require clear thinking for maximum efficiency.

One element that should not be neglected is the occurrence of such problems in public questions which must be decided by the votes of the whole people. The tariff, rates of postage and express, freight rates, regulation of insurance rates, income taxes, inheritance taxes, and many other public questions involve relationships between quantities — for example, between the rate of income taxation and the amount of the income — that require habits of functional thinking for intelligent decisions. The training in such habits of thinking is therefore a vital element toward the creation of good citizenship.

It is believed that transfer of training does operate between such topics as those suggested in the body of this chapter and those just mentioned, because of the existence

of such identical or common elements, whereas the transfer of the training given by courses in mathematics that do not emphasize functional relationships might be questionable.

While this account of the functional character of certain topics in geometry and in algebra makes no claim to being exhaustive, the topics mentioned will suggest others of like character to the thoughtful teacher. It is hoped that sufficient variety has been mentioned to demonstrate the existence of functional ideas throughout elementary algebra and geometry. The committee feels that if this is recognized, algebra and geometry can be given new meaning to many children, and indeed to many educators, and that all students will be better able to control the actual relations which they meet in their own lives.

VIII

TERMS AND SYMBOLS IN ELEMENTARY MATHEMATICS[1]

A. Limitations imposed by the committee upon its work. The committee feels that in dealing with this subject it should explicitly recognize certain general limitations, as follows:

1. No attempt should be made to impose the phraseology of any definition, although the committee should state clearly its general views as to the meaning of disputed terms.

2. No effort should be made to change any well-defined current usage unless there is a strong reason for doing so, which reason is supported by the best authority, and, other things being substantially equal, the terms used should be international. This principle excludes the use of all individual efforts at coining new terms except under circumstances of great urgency. The individual opinions of the members, as indeed of any teacher or body of teachers, should have little weight in comparison with general usage if this usage is definite. If an idea has to be expressed so often in elementary mathematics that it becomes necessary to invent a single term or symbol for the purpose, this invention is necessarily the work of an individual; but it is highly desirable, even in this case,

[1] The first draft of this chapter was prepared by a subcommittee consisting of David Eugene Smith (chairman), W. W. Hart, H. E. Hawkes, E. R. Hedrick, and H. E. Slaught.

that it should receive the sanction of wide use before it is adopted in any system of examinations.

3. On account of the large number of terms and symbols now in use, the recommendations to be made will necessarily be typical rather than exhaustive.

I. Geometry

B. Undefined terms. The committee recommends that no attempt be made to define, with any approach to precision, terms whose definitions are not needed as parts of a proof.

Especially is it recommended that no attempt be made to define precisely such terms as *space*, *magnitude*, *point*, *straight line*, *surface*, *plane*, *direction*, *distance*, and *solid*, although the significance of such terms should be made clear by informal explanations and discussions.

C. Definite usage recommended. It is the opinion of the committee that the following general usage is desirable:

1. *Circle* should be considered as the curve; but where no ambiguity arises, the word "circle" may be used to refer either to the curve or to the part of the plane inclosed by it.

2. *Polygon* (including *triangle*, *square*, *parallelogram*, and the like) should be considered, by analogy to a circle, as a closed broken line; but where no ambiguity arises, the word polygon may be used to refer either to the broken line or to the part of the plane inclosed by it. Similarly, *segment of a circle* should be defined as the figure formed by a chord and either of its arcs.

3. *Area of a circle* should be defined as the area (numerical measure) of the portion of the plane inclosed by the

circle. *Area of a polygon* should be treated in the same way.

4. *Solids.* The usage above recommended with respect to plane figures is also recommended with respect to solids. For example, *sphere* should be regarded as a surface, its volume should be defined in a manner similar to the area of a circle, and the double use of the word should be allowed where no ambiguity arises. A similar usage should obtain with respect to such terms as *polyhedron*, *cone*, and *cylinder*.

5. *Circumference* should be considered as the length (numerical measure) of the circle (line). Similarly, *perimeter* should be defined as the length of the broken line which forms a polygon; that is, as the sum of the lengths of the sides.

6. *Obtuse angle* should be defined as an angle greater than a right angle and less than a straight angle, and should therefore not be defined merely as an angle greater than a right angle.

7. The term *right triangle* should be preferred to "right-angled triangle," this usage being now so standardized in this country that it may properly be continued in spite of the fact that it is not international. Similarly for *acute triangle*, *obtuse triangle*, and *oblique triangle*.

8. Such English plurals as *formulas* and *polyhedrons* should be used in place of the Latin and Greek plurals. Such unnecessary Latin abbreviations as *Q.E.D.* and *Q.E.F.* should be dropped.

9. The definitions of *axiom* and *postulate* vary so much that the committee does not undertake to distinguish between them.

D. Terms made general. It is the recommendation

of the committee that the modern tendency of having terms made as general as possible should be followed. For example:

1. *Isosceles triangle* should be defined as a triangle having two equal sides. There should be no limitation to two and only two equal sides.

2. *Rectangle* should be considered as including a square as a special case.

3. *Parallelogram* should be considered as including a rectangle, and hence a square, as a special case.

4. *Segment* should be used to designate the part of a straight line included between two of its points as well as the figure formed by an arc of a circle and its chord, this being the usage generally recognized by modern writers.

E. Terms to be abandoned. It is the opinion of the committee that the following terms are not of enough consequence in elementary mathematics at the present time to make their recognition desirable in examinations, and that they serve chiefly to increase the technical vocabulary to the point of being burdensome and unnecessary:

1. *Antecedent* and *consequent.*

2. *Third proportional* and *fourth proportional.*

3. *Equivalent.* An unnecessary substitute for the more precise expressions "equal in area" and "equal in volume," or (where no confusion is likely to arise) for the single word "equal."

4. *Trapezium.*

5. *Scholium, lemma, oblong, scalene triangle, sect, perigon, rhomboid* (the term "oblique parallelogram" being sufficient), and *reflex angle* (in elementary geometry).

6. Terms like *flat angle*, *whole angle*, and *conjugate angle* are not of enough value in an elementary course to make it desirable to recommend them.

7. *Subtend*, a word which has no longer any etymological meaning to most students and teachers of geometry. While its use will naturally continue for some time to come, teachers may safely incline to such forms as the following: "In the same circle equal arcs *have* equal chords."

8. *Homologous*, the less technical term "corresponding" being preferable.

9. Guided by principle A 2 and its interpretation, the committee advises against the use of such terms as the following: *Angle-bisector*, *angle-sum*, *consecutive interior angles*, *supplementary consecutive exterior angles*, *quader* (for rectangular solid), *sect*, *explement*, *transverse angles*.

10. It is unfortunate that it still seems to be necessary to use such a term as *parallelepiped*, but we seem to have no satisfactory substitute. For rectangular parallelepiped, however, the use of *rectangular solid* is recommended. If the terms were more generally used in elementary geometry it would be desirable to consider carefully whether better ones could not be found for the purposes than *isoperimetric*, *apothem*, *icosahedron*, and *dodecahedron*.

F. Symbols in elementary geometry. It should be recognized that a symbol like $\perp$ is merely a piece of shorthand designed to afford an easy grasp of a written or printed statement. Many teachers and a few writers make an extreme use of symbols, coining new ones to meet their own views as to usefulness, and this practice

is naturally open to objection.[2] There are, however, certain symbols that are international and certain others of which the meaning is at once apparent and which are sufficiently useful and generally enough recognized to be recommended.

For example, the symbols for triangle, $\triangle$, and circle, $\odot$, are international, although used more extensively in the United States than in other countries. Their use, with their customary plurals, is recommended.

The symbol $\perp$, generally read as representing the single word "perpendicular" but sometimes as standing for the phrase "is perpendicular to," is fairly international and the meaning is apparent. Its use is therefore recommended. On account of such a phrase as "the $\perp AB$," the first of the above readings is likely to be the more widely used, but in either case there is no chance for confusion.

The symbol $\parallel$ for "parallel" or "is parallel to" is fairly international and is recommended.

The symbol $\sim$ for "similar" or "is similar to" is international and is recommended.

The symbols $\cong$ and $\equiv$ for "congruent" or "is congruent to" both have a considerable use in this country. The committee feels that the former, which is fairly international, is to be preferred because it is the more distinctive and suggestive.

The symbol $\angle$ for "angle" is, because of its simplicity, coming to be generally preferred to any other and is therefore recommended.

[2] This is not intended to discourage the use of algebraic methods in the solution of such geometric problems as lend themselves readily to algebraic treatment.

Since the following terms are not used frequently enough to render special symbols of any particular value the world has not developed any that have general acceptance and there seems to be no necessity for making the attempt: Square, rectangle, parallelogram, trapezoid, quadrilateral, semicircle.

The symbol $\widehat{AB}$ for "arc AB" cannot be called international. While the value of the symbol $\frown$ in place of the short word *arc* is doubtful, the committee sees no objection to its use.

The symbol $.\dot{}.$ for "therefore" has a value that is generally recognized, but the sumbol $\dot{}.\dot{}$ for "since" is used so seldom that it should be abandoned.

With respect to the lettering of figures, the committee calls attention for purposes of general information to a convenient method, found in certain European and in some American textbooks, of lettering triangles: Capitals represent the vertices, corresponding small letters represent opposite sides, corresponding small Greek letters represent angles, and the primed letters represent the corresponding parts of a congruent or similar triangle. This permits of speaking of α (alpha) instead of "angle A," and of "small a" instead of BC. The plan is by no means international with respect to the Greek letters. The committee is prepared, however, to recommend it with the optional use of the Greek forms.

In general, it is recommended that a single letter be used to designate any geometric magnitude whenever there is no danger of ambiguity. The use of numbers alone to designate magnitudes should be avoided by the use of such forms A_1, A_2, . . .

With respect to the symbolism for limits, the committee

calls attention to the fact that the symbol $\doteq$ is a local one, and that the symbol $\rightarrow$ (for "tends to") is both international and expressive and has constantly grown in favor in recent years. Although the subject of limits is not generally treated scientifically in the secondary school, the idea is mentioned in geometry and a symbol may occasionally be needed.

While the teacher should be allowed freedom in the matter, the committee feels that it is desirable to discourage the use of such purely local symbols as the following:

$\stackrel{\circ}{=}$ for "equal in degrees,"

ass for "two sides and an angle adjacent to one of them," and

sas for "two sides and the included angle."

G. Terms not standardized. At the present time there is not sufficient agreement upon which to base recommendations as to the use of the term *ray* and as to the value of terms like *coplanar*, *collinear*, and *concurrent* in elementary work. Many terms, similar to these, will gradually become standardized or else will naturally drop out of use.

II. Algebra and Arithmetic

H. Terms in algebra. 1. With respect to equations the committee calls attention to the fact that the classification according to degree is comparatively recent and that this probably accounts for the fact that the terminology is so unsettled. The Anglo-American custom of designating an equation of the first degree as a *simple equation* has never been satisfactory, because the term has no real significance. The most nearly international

terms are *equation of the first degree* (or "first degree equation") and *linear equation.* The latter is so brief and suggestive that it should be generally adopted.

2. The term *quadratic equation* (for which the longer term "second degree equation" is an unnecessary synonym, although occasionally a convenient one) is well established. The terms *pure quadratic* and *affected quadratic* signify nothing to the pupil except as he learns the meaning from a book, and the committee recommends that they be dropped. Terms more nearly in general use are *complete quadratic* and *incomplete quadratic.* The committee feels, however, that the distinction thus denoted is not of much importance and believes that it can well be dispensed with in elementary instruction.

3. As to other special terms, the committee recommends abandoning, so far as possible, the use of the following: *Aggregation* for grouping; *vinculum* for bar; *evolution* for finding roots, as a general topic; *involution* for finding powers; *extract* for find (a root); *absolute term* for constant term; *multiply an equation, clear of fractions, cancel* and *transpose,* at least until the significance of the terms is entirely clear; *aliquot part* (except in commercial work).

4. The committee also advises the use of either *system of equations* or *set of equations* instead of "simultaneous equations," in such an expression as "solve the following set of equations," in view of the fact that at present no well established definite meaning attaches to the term "simultaneous."

5. The term *simplify* should not be used in cases where there is possibility of misunderstanding. For purposes of computation, for example, the form $\sqrt{8}$ may be simpler

than the form $2\sqrt{2}$, and in some cases it may be better to express $\sqrt{3/4}$ as $\sqrt{0.75}$ instead of $\frac{1}{2}\sqrt{3}$. In such cases, it is better to give more explicit instructions than to use the misleading term "simplify."

6. The committee regrets the general uncertainty in the use of the word *surd*, but it sees no reasonable chance at present of replacing it by a more definite term. It recognizes the difficulty generally met by young pupils in distinguishing between *coefficient* and *exponent*, but it feels that it is undesirable to attempt to change terms which have come to have a standardized meaning and which are reasonably simple. These considerations will probably lead to the retention of such terms as *rationalize*, *extraneous root*, *characteristic*, and *mantissa*, although in the case of the last two terms "integral part" and "fractional part" (of a logarithm) would seem to be desirable substitutes.

7. While recognizing the motive that has prompted a few teachers to speak of "positive x" instead of "plus x," and "negative y" instead of "minus y" the committee feels that attempts to change general usage should not be made when based upon trivial grounds and when not sanctioned by mathematicians generally.

I. Symbols in algebra. The symbols in elementary algebra are now so well standardized as to require but few comments in a report of this kind. The committee feels that it is desirable, however, to call attention to the following details:

1. Owing to the frequent use of the letter x, it is preferable to use the center dot (a raised period) for multiplication in the few cases in which any symbol is necessary. For example, in a case like $1\cdot2\cdot3\cdot\ldots(x-1)\cdot x$, the center

dot is preferable to the symbol $\times$; but in cases like $2a(x - a)$ no symbol is necessary. The committee recognizes that the period (as in $a.b$) is more nearly international than the center dot as in $(a \cdot b)$; but inasmuch as the period will continue to be used in this country as a decimal point, it is likely to cause confusion, to elementary pupils at least, to attempt to use it as a symbol for multiplication.

2. With respect to division, the symbol $\div$ is purely Anglo-American, the symbol : serving in most countries for division as well as ratio. Since neither symbol plays any part in business life, it seems proper to consider only the needs of algebra, and to make more use of the fractional form and (where the meaning is clear) of the symbol /, and to drop the symbol $\div$ in writing algebraic expressions.

3. With respect to the distinction between the use of $+$ and $-$ as symbols of operation and as symbols of direction, the committee sees no reason for attempting to use smaller signs for the latter purpose, such an attempt never having received international recognition, and the need of two sets of symbols not being sufficient to warrant violating international usage and burdening the pupil with this additional symbolism.

4. With respect to the distinction between the symbols $\equiv$ and $=$ as representing respectively identity and equality, the committee calls attention to the fact that, while the distinction is generally recognized, the consistent use of the symbols is rarely seen in practice. The committee recommends that the symbol $\equiv$ be not employed in examinations for the purpose of indicating identity. The teacher, however, should use both symbols if desired.

5. With respect to the root sign, $\surd$, the committee recognizes that convenience of writing assures its continued use in many cases instead of the fractional exponent. It is recommended, however, that in algebraic work involving complicated cases the fractional exponent be preferred. Attention is also called to the fact that the convention is quite generally accepted that the symbol $\sqrt{a}$ (a representing a positive number) means only the positive square root and that the symbol $\sqrt[n]{a}$ means only the principal nth root, and similarly for $a^{\frac{1}{2}}$, $a^{\frac{1}{n}}$. The reason for this convention is apparent when we come to consider the value of $\sqrt{4}+\sqrt{9}+\sqrt{16}+\sqrt{25}$. This convention being agreed to, it is improper to write $x=\sqrt{4}$, as the complete solution of $x^2-4=0$, but the result should appear as $x=\pm\sqrt{4}$. Similarly, it is not in accord with the convention to write $\sqrt{4}=\pm 2$, the conventional form being $\pm\sqrt{4}=\pm 2$; and for the same reason it is impossible to have $\sqrt{(-1)^2}=\pm 1$ since the symbol refers only to a positive root. These distinctions are not matters to be settled by the individual opinion of a teacher or a local group of teachers; they are purely matters of convention as to notation, and the committee simply sets forth, for the benefit of teachers, this statement as to what the convention seems to be among the leading writers of the world at the present time.

6. When imaginaries are used, the symbol i should be employed instead of $\sqrt{-1}$ except possibly in the first presentation of the subject.

7. As to the factorial symbols $5\,!$ and $\underline{|5}$, to represent $5\cdot4\cdot3\cdot2\cdot1$, the tendency is very general to abandon the second one, probably on account of the difficulty of print-

ing it, and the committee so recommends. This question is not, however, of much importance in the general courses in the high school.

8. With respect to symbols for an unknown quantity there has been a noteworthy change within a few years. While the Cartesian use of x and y will doubtless continue for two general unknowns, the recognition that the formula is, in the broad use of the term, a central feature of algebra has led to the extended use of the initial letter. This is simply illustrated in the direction to solve for r the equation $A = \pi r^2$. This custom is now international and should be fully recognized in the schools.

9. The committee advises abandoning the double colon (: :) in proportion, and the symbol $\propto$ in variation, both of these symbols being now practically obsolete.

J. Terms and symbols in arithmetic. 1. While it is rarely wise to attempt to abandon suddenly the use of words that are well established in our language, the committee feels called upon to express regret that we still require very young pupils, often in the primary grades, to use such terms as *subtrahend*, *addend*, *minuend* and *multiplicand*. Teachers themselves rarely understand the real significance of these words, nor do they recognize that they are comparatively modern additions to what used to be a much simpler vocabulary in arithmetic. The committee recommends that such terms be used, if at all, only after the sixth grade.

2. Owing to the uncertainty attached to such expressions as "to three decimal places," "to thousandths," "correct to three decimal places," "correct to the nearest thousandth," the following usage is recommended: When used to specify accuracy in computation, the four expres-

sions should be regarded as identical. The expression "to three decimal places" or "to thousandths" may be used in giving directions as to the extent of a computation. It then refers to a result carried only to thousandths, without considering the figure of ten-thousandths; but it should be avoided as far as possible because it is open to misunderstanding. As to the term, "significant figure," it should be noted that 0 is always significant except when used before a decimal fraction to indicate the absence of integers or, in general, when used merely to locate the decimal point. For example the zeros italicized in the following are "significant," while the others are not: 0.5, 9.5*0*, 1*0*2, 3*0*,200. Further, the number 2396, if expressed correct to three significant figures, would be written 24*0*0.[3] It should be noted that the context or the way in which a number has been obtained is sometimes the determining factor as to the significance of a 0.

3. The pupil in arithmetic needs to see the work in the form in which he will use it in practical life outside the schoolroom. His visualization of the process should therefore not include such symbols as +, −, ×, ÷, which are helpful only in writing out the analysis of a problem or in the printed statement of the operation to be performed. Because of these facts the committee recommends that only slight use be made of these symbols in the written work of the pupil, except in the analysis of problems. It recognizes, however, the value of such symbols in printed directions and in these analyses.

[3] The italicizing of significant zeros is here used merely to make clear the committee's meaning: The device is not recommended for general adoption.

III. General Observations and Recommendations

K. General Observations. The committee desires also to record its belief in two or three general observations.

1. It is very desirable to bring mathematical writing into closer touch with good usage in English writing in general. That we have failed in this particular has been the subject of frequent comment by teachers of mathematics as well as by teachers of English. This is all the more unfortunate because mathematics may be and should be a genuine help toward the acquisition of good habits in the speaking and writing of English. Under present conditions, with a style that is often stilted and in which undue compression is evident, we do not offer to the student the good models of English writing of which mathematics is capable, nor indeed do we always offer good models of thought processes. It is to be feared that many teachers encourage the use of a kind of vulgar mathematical slang when they allow such words as "tan" and "cos," for tangent and cosine, and habitually call their subject by the title "math."

2. In the same general spirit the committee wishes to observe that teachers of mathematics and writers of text-books seem often to have gone to an extreme in searching for technical terms and for new symbols. The committee expresses the hope that mathematics may retain, as far as possible, a literary flavor. It seems perfectly feasible that a printed discussion should strike the pupil as an expression of reasonable ideas in terms of reasonable English forms. The fewer technical terms we introduce, the less is the subject likely to give the impression of being difficult and a mere juggling of words and symbols.

3. While recognizing the claims of euphony, the fact that a word like "historic" has a different meaning from "historical" and that confusion may occasionally arise if "arithmetic" is used as an adjective with a different pronunciation from the noun, the committee advises that such forms as geometric be preferred to geometrical. This is already done in such terms as analytic geometry and elliptic functions, and it seems proper to extend the custom to include *arithmetic*, *geometric*, *graphic*, and the like.

L. General recommendations. 1. In view of the desirability of a simplification of terms used in elementary instruction, and of establishing international usage so far as is reasonable, the committee recommends that the subject of this report be considered by a committee to be appointed by Section IV of the next International Congress of Mathematicians, such committee to contain representatives of at least the recognized international languages admitted to the meetings.

2. The committee suggests that examining bodies, contributors to mathematical journals, and authors of textbooks endeavor to follow the general principles formulated in this report.

APPENDIX

1. THE PRESENT STATUS OF DISCIPLINARY VALUES IN EDUCATION (Extract)

By VEVIA BLAIR, Horace Mann School for Girls

[Editor's Note: The following is an extract from Chapter IX of the complete report. Miss Blair made a very thorough study of the publications of psychologists on the subject of "formal discipline" and on the basis of this study formulated the inferences with which the following extract begins. These inferences were then submitted to forty leading psychologists of this country for their consideration and comment. Of the twenty-seven who replied three did not give permission to use their names. The names of the remaining twenty-four together with their replies in full were published in the complete report. Here the space available permits the publication only of Sections 3 and 4 of Miss Blair's investigation, giving respectively the "Formulation of Inferences" and "The Judgment of Psychologists and General Conclusions."]

3. A Formulation of Inferences

From the general considerations listed above the following seven statements were formulated and submitted to psychologists for their consideration. The psychologists were asked for any further information they would be willing to give, and it was hoped that a free expression of opinion on their part in regard to any debatable point would tend to obviate some of the objectionable features of a questionnaire.

I. Transfer of training is an established fact, and may be positive, negative, or zero.

II. The true amount of transfer from one field to another has not yet been found by experiment, on account of one or more of the following handicaps:

(*a*) *The maturity or previous training of the subjects tested.*

(*b*) *The absence from the training of the factors most favorable to transfer.*

(*c*) *The inadequacy of the tests to measure the traits sought.*

III. It is a reasonable inference that a substantial amount of transfer to some related field would be found by an adequate test, if

in teaching children, emphasis were placed upon the trait which the selected subject was most capable of developing, and if the factors controlling transfer were present in the training.

IV. Even if no great amount of transfer of training to any one field should ever be found by experiment, it would still be true that if small amounts of transfer of a valuable trait extended to a large number of fields, the sum total of all these small amounts would be a very valuable educational asset.

V. Negative transfer or interference may take place when in the training of a certain trait, auxiliary habits are cultivated, which have to be broken down before the trait can function in a new situation.

VI. Zero transfer may occur when the habits tending to interfere, and those tending to transfer, just offset each other.

VII. There are elements of situations so fundamental in their nature that they occur again and again in connection with almost anything else. Special training with these elements has general value.

4. The Judgment of Psychologists and General Conclusions

In the accompanying table an attempt has been made to show roughly the nature of the responses to each statement. The figures in the body of the table indicate numbers of men; totals are given both in numbers and in percentages. Questions misunderstood are listed as omitted.

The general conclusions drawn from this table and from the comments which some of the psychologists added to their answers may be stated as follows:

1. The two extreme views for and against disciplinary values practically no longer exist. As the question now stands, as transfer of training, the psychologists quoted here almost unanimously agree that transfer does exist.

2. A large majority agree that there is a possibility of negative transfer, and of zero transfer, caused by interference effects. Professor Thorndike is of the opinion that negative

Statement	Yes	Yes Qualified	Doubtful	No	Omitted	Total
I.......	21	2			1	24
II.......	19	2		1	2	24
II a.....	13	2	2	2	5	24
II b.....	17	1	2	2	2	24
II c.....	15	2	2	2	3	24
III.......	11	3	2	1	7	24
IV.......	11	5	1	2	5	24
V.......	20	1	3			24
VI.......	17	5			2	24
VII.......	21	1			2	24
Total.....	165	24	12	10	29	240
Per cents..	69	10	5	4	12	100

transfer is comparatively rare and can be avoided by proper methods of training.

3. Very few if any experiments have shown the full amount of transfer between the fields chosen for investigation. The reason for this is to be found in the imperfections of the experimental setting. Of the defects mentioned in Statement II, absence from the training of the factors most favorable to transfer is the one most generally considered important, though strong emphasis is placed in several instances upon the inadequacy of the tests.

4. The amount of transfer in any case where transfer is admitted at all, is very largely dependent upon methods of teaching. This is probably the strongest note struck by the psychologists in their comments. Twelve of them out of twenty-four make some explicit reference to the matter.

5. A majority of the psychologists, judging from their remarks in connection with Statements III, IV, and VII seem to believe that, with certain restrictions, transfer of training is a valid aim in teaching.

6. Transfer is most evident with respect to general elements — ideas, attitudes, and ideals. These act in many instances as the carriers in transfer. Often they form the common element so generally held to be the *sine qua non* of transfer. Twenty-two of the twenty-four psychologists ex-

press the opinion that special training in connection with these elements has general value; and one of them, Professor Sanford, adds: "This I believe to be true, and to be the basis of the generally held belief among practical teachers of the existence and value of 'formal training.'"

[The names of the psychologists, whose replies are published in full in the complete report are: J. R. Angell, W. C. Bagley, H. V. Bingham, E. G. Boring, S. S. Colvin, G. O. Ferguson, V. A. C. Henmon, Joseph Jastrow, C. H. Judd, G. T. Ladd, R. M. Ogden, W. P. Pilsbury, E. C. Sanford, W. D. Scott, C. E. Seashore, Daniel Starch, E. K. Strong, L. M. Terman, E. L. Thorndike, H. C. Warren, M. F. Washburn, J. B. Watson, R. S. Woodworth, R. M. Yerkes.]

2. THE TRAINING OF TEACHERS OF MATHEMATICS
(Extracts)

By RAYMOND CLARE ARCHIBALD, Brown University

[Editor's Note: The following are two brief extracts from the extensive report by Professor Archibald published as Chapter XIV of the complete report. The first is taken from Section I of this chapter and is intended to give a summary of conditions in foreign countries. The second is Section IV in full and relates to a tentative standard and program for the training of teachers of mathematics for secondary schools in this country.]

1. The Training of Teachers of Mathematics for Secondary Schools in Foreign Countries

In 1912 about 150 reports[1] regarding the teaching of mathematics in seventeen foreign countries were presented to the International Congress of Mathematicians at Cambridge, England. With these as a basis the writer prepared an extensive report, published by the Bureau of Education, Washington, D.C., in 1918, on *The Training of Teachers of Mathematics for the Secondary Schools of the Countries represented in*

[1] The International Commission on the Teaching of Mathematics stated in July, 1921 (Enseignement Mathématique), that it had published 187 volumes or fascicules which it listed, containing 310 reports of 13,565 pages.

the International Commission on the Teaching of Mathematics. For each country considered, rather detailed information and full bibliographies were given with reference to the training of teachers to serve in better secondary schools for boys.

A summary of the material thus set forth is here adapted for an introductory chapter which may be suggestive to those who thoughtfully consider in what way desirable reforms in connection with our secondary schools may be brought about. That the contrast in standards and ideals may be more marked, this summary is placed in immediate contiguity to some indications of corresponding conditions in the United States.

In instituting such a comparison it should be borne in mind that there is no consensus of opinion throughout the world as to the periods to be devoted to primary, or to secondary, education. Most foreign countries agree that twelve or thirteen years normally lead to a university. But while the United States and Australia hold that only four years [2] should be devoted to secondary education, seven countries (Austria, Finland, Hungary, Italy, Japan, Roumania, and Russia) set aside eight years, and three countries (France, with her two extra years for the mathematical or military or naval or engineering specialist, Germany, and Sweden) nine years.

In contrast to eight years of elementary education [2] in the United States, France and Germany consider that the best results are obtained when even the three or four years allotted to primary instruction are given in connection with secondary schools. It is then not surprising to find that there is great difference between the scholastic equipment of students coming from these two general types of school. The graduates of the classe de mathématiques spéciales in France, or of the German Gymnasium are about on a par with the youth who has finished his junior year in one of the better American colleges. And in other countries also, like Denmark, Japan, and

[2] The recent movement towards the establishment of junior high schools tends to reduce the elementary period to six years and to extend the secondary period to six years.

Sweden, the graduate of a higher secondary school has covered considerable of the work of which the equivalent is offered in colleges in the United States. It is only in the light of such considerations that the full force of, say, Sweden's requirement of ten or eleven years of preparation before a graduate of a Gymnasium may return as a regularly appointed professor can be adequately appreciated.

. . . .

Summary

These very brief sketches of the conditions in foreign countries can suggest only in imperfect outline some outstanding features as to the manner in which prospective teachers of mathematics in the secondary schools are trained. After leaving the secondary schools the training is derived, broadly speaking, from two sources: I. Courses in a university or a similar higher institution; and II. Professional training.

I. All the countries require some university training on the part of candidates for appointment as secondary school teachers.[3] The maximum requirements are in Denmark and Netherlands, each 6 years, and in Sweden, about 8 years. On the other hand, for minor positions in the athénées of Belgium and regular positions in Canton Vaud, Switzerland (where most Cantons require 4 or 5 years), only 2 or 2½ years of attendance at a university are compulsory. The complete record is as follows: Australia (Victoria and New South Wales), 3 years; Austria, 3½ to 4; Belgium, 4 to 5, 2 for minor positions in athénées; Denmark, 6; England, 3 to 4; Finland, 4 to 5; France, not less than 3, in addition to 2 years in classes de mathématiques spéciales; Germany, 3 to 4, but rarely 3 and often 5 are taken; Hungary, 4; Italy, 4 to 5; Spain, 5, for lower positions, 4; Sweden, usually 8; Switzerland, 4 to 5 years for the most part, in one or two cases 2 to 3.

Let us consider an example to bring out more clearly the

[3] So far as this statement concerns Japan, reference is made to the higher middle schools.

implications of these statements. Since the future mathematical teacher entering a German university is about on a scholastic par with the student who has finished the junior year at a college in the United States, we may state, roughly, that the German teacher has generally had at least three years more of scientific training than the prospective teacher going out as a college graduate in the United States. Moreover, in contrast, his training has, as a rule, been based upon a secondary education relentlessly thorough, and conducted by those who are masters of their professions; in the university his line of work has been very specialized. Such specialization is characteristic in the case of most of the foreign countries considered above in connection with the preparation of their teachers for secondary schools.

The lists of courses taken by such students indicate how the majority of the universities emphasize instruction in such subjects as plane and solid analytic geometry, calculus, functions of a real variable, differential equations, differential geometry, descriptive geometry, mechanics, and physics. Projective geometry is prominent in such countries as Belgium, Denmark, France, Italy, and Spain; various topics of higher algebra in Hungary, Italy, the Netherlands, Roumania and Russia; foundations of mathematics in Austria, Belgium, Japan, and Spain; and history of mathematics in Belgium and Denmark.

It is to be borne in mind that mere names of courses mentioned in connection with different countries do not, without much elaboration, convey a very definite idea. Calculus as a requirement for the prospective teacher in Japan is vastly different from that in Belgium where the equivalent of de la Vallée Poussin's treatise on analysis may be the basis for part of the first two years of work in a university. What is given in our first and second courses of calculus in American colleges is covered in special secondary courses of France, so that the student at a university starts in with courses in analysis and differential geometry such as are outlined in the first volume and part of the second of Goursat's treatise on analysis.

II. In addition to attendance at universities, some countries require professional training. Australia (Victoria and New South Wales), England (generally), Finland, Roumania, and some states of Germany each require one year (it is only in theory that Austria requires a year); other states of Germany and Denmark require two years each; in addition to a year in a seminary, Sweden requires two years of probation as teacher before regular appointment and in a similar way Hungary requires three; and in Italy after four years of trial a teacher may be dropped. In France the professional training may possibly be estimated at half a year. In six countries no professional training is made compulsory. These countries are: Belgium, Japan (in higher middle schools), the Netherlands, Russia, Spain, and Switzerland. Details regarding the nature of professional training in different countries, and as to its comparative efficiency, may be found in the report referred to in the first paragraph of this section, page 126.

. . . .

IV. A Tentative Standard for the Training of Teachers of Mathematics, and Courses Primarily Intended for Such Teachers

We have noted the very wide divergence in the United States of standards regarding the certification of teachers of mathematics in senior high schools. Is it possible at the present time to set up *any* standard that would be generally acceptable? The answer must be unqualifiedly in the negative. Indeed, in States where the standards are legally very high, it has recently been found extremely difficult to secure a sufficient number of teachers with the desired qualifications. (See the remarks of National Committee, pp. 23 ff.)

There is, however, a general belief that, with public realization of the fundamental importance of secondary education, the economic condition of the secondary school teacher will materially improve and the attractions of his work essentially

increase. Looking forward then to a happier period, it seems decidedly advisable to formulate a standard for certification which would be generally desirable if it could be put into effect — a formulation suggestive to colleges and universities throughout the land in raising standards to a reasonable status, even though this be far below that of several European countries. It will be observed that the standard here suggested is practically equivalent to that demanded by one State of the Union acting in coöperation with its state university. It is hoped, therefore, that other States will be the more ready to consider the adoption of a policy approximating the following tentative ideals:

To receive *permanent* appointment as a teacher of mathematics in a senior high school a candidate should satisfy the following requirements, or their equivalent:

1. Graduation from a standard four year college, or university, or from an institution offering courses of at least equal difficulty and educational value.
2. Credit for at least the following mathematical courses (given by teachers of mathematics in colleges or universities):
   (*a*) Plane and spherical trigonometry;
   (*b*) Plane analytic geometry and the elements of analytic geometry of three dimensions;
   (*c*) College algebra (1 semester [4]);
   (*d*) Differential and integral calculus, with applications to geometry and mechanics (3 semesters);
   (*e*) Synthetic projective geometry (1 semester);
   (*f*) Scientific training in geometry (2 semesters) — (i) *first semester:* Text, J. Petersen's *Methods and Theories for the Solution of Problems of Geometrical Constructions* [5] and accompanying lectures to present the history of the famous problems, and the history of elementary geo-

[4] Here, and in what follows, it shall be supposed that the term "semester" implies about 45 lectures or recitations — 3 hours a week for 15 weeks.

[5] Specific books are mentioned, here and later, in order to give definiteness to the suggestions. It is to be understood, however, that other texts are not necessarily regarded as less desirable.

metry; (ii) *second semester:* Texts, F. Klein's *Famous Problems of Elementary Geometry* (except chapters on transcendence of e and π), J. W. Young's *Lectures on Fundamental Concepts of Algebra and Geometry* (selected chapters), and, for half the semester, J. Hadamard's *Leçons de Géométrie Elémentaire*, vol. 1, *Géométrie plane*, vol. 2, *Géométrie dans l'espace*, especially the chapters on proportional lines, areas, regular polygons, dihedral and polyhedral angles, polyhedra, cylinders, cones, and spheres, and the notes on Euclid's postulate, notions of area, definitions of volumes, regular polyhedra, and groups of rotations.

(*g*) Scientific training in algebra (2 semesters) — Lectures on the history of elementary algebra and topics from the following texts: J. W. Young's *Lectures on Fundamental Concepts of Algebra and Geometry* (selected chapters), Fine's *College Algebra* (especially the parts on numbers, fundamental theorem of algebra, h.c.f. and l.c.m., symmetric functions, convergence and divergence of series), F. Klein's *Famous Problems of Elementary Geometry* (chapters on transcendence of e and π).

3. Credit for at least the following scientific courses: Theoretical and practical physics (3 semesters), chemistry (2 semesters).
4. Credit for at least the following theoretical professional courses (4 semesters; given by teachers of education): History of education, Principles of education, methods of teaching (including the teaching of elementary algebra and geometry), educational psychology, organization and function of secondary education.
5. Satisfactory performance of the duties of a teacher of mathematics in a secondary school for a period of not less than 10 year, or 20 semester, hours. It is considered by many competent authorities that the most satisfactory conditions for this practical professional training are in connection with a year of postgraduate work in a school of education organized so that continuous directed teaching of classes in public schools throughout the year is available to students.

It is believed that college semester-courses in rational solid geometry, descriptive geometry, analytic projective geometry, theory

of statistics, mathematics of investment, surveying, practical and descriptive astronomy, and in as many other mathematical topics as possible, are also desirable.

It is generally conceded as further desirable that the prospective teacher should have studied during the college course the following subjects: History, economics, sociology, political science, general psychology, philosophy, and ethics.

Finally, it is suggested to state departments of education that the best interests of its teachers, and their charges, would be conserved, if during the first twenty years of service, an appreciable portion of periodic salary increases of the teacher were made dependent upon continued scientific development (for example, by means of summer-school courses), and upon active coöperation with colleagues in promoting the interests of mathematics and the ideals and purposes of mathematical organizations.

3. THE TESTING MOVEMENT IN MATHEMATICS

BY C. B. UPTON, TEACHERS COLLEGE, COLUMBIA UNIVERSITY

The testing movement in mathematics began about twenty years ago when Courtis first published his Test in Arithmetic. Since that time numerous other tests in arithmetic, algebra, and geometry have been developed. A study of these tests indicates that the following types are now available.

Speed tests. This type of test, best illustrated by the
Courtis Research Test in Arithmetic, or the original
Rugg-Clark Test in Algebra, aims to see how many ex- 837
amples of equal difficulty a pupil can do in a given time. 882
For example, he may be asked to add as many ex- 959
amples, each similar to the one at the right, as he can in 603
8 minutes. 118

Scales or power tests. Mathematical scales, illus- 781
trated by the Woody Arithmetic Scale, or the Hotz Alge- 756
bra Scale, are so constructed that they begin with most 222
simple exercises and gradually progress to those that are 525

more difficult. Such tests, also called "power" tests, serve in certain ways for diagnostic purposes. For example, if a pupil works nine examples in such a test and fails on the tenth, it is usually because some new difficulty is presented in the tenth which he has not yet mastered. Hence the teacher has a way of knowing where the pupil's weakness lies.

Diagnostic tests. The mathematical scales just mentioned are one form of diagnostic test. Another form, illustrated by one of the diagnostic tests in arithmetic, presents in subtraction, for example, problems of various types of difficulty in the subtraction of whole numbers and fractions, though the problems are not all arranged in order of difficulty, as in the Woody Scale. The aim of such tests is to discover the types of examples with which the pupil is not familiar.

Achievement tests. Each of the types of tests mentioned above provides a certain measure of achievement. To these should be added certain tests which are really a new type of examination in mathematics, such as the Schorling-Sanford Achievement Test in Plane Geometry.

Reasoning or problem tests. Such tests aim to examine a pupil's ability to work the usual concrete problems in arithmetic or algebra, whereas the tests described above have been confined to the abstract operations.

Tests of mathematical ability. Tests of this type are in some respects similar to an intelligence examination. The Rogers Test of Mathematical Ability, for example, aims to determine in advance a pupil's innate mathematical ability in order to guide him in his future study of mathematics in the high school.

Tests such as those described above have contributed considerably to the efficiency of the teaching of mathematics in both elementary and secondary schools by fixing standards of accomplishment; by giving a basis for comparing the work of one class or of one school with another; by directing the attention of teachers to certain large essentials of the subject,

thus keeping them from neglecting certain topics and overemphasizing others; by giving definite evidence concerning the relative difficulty of certain topics; and by providing teachers with a measure of the efficiency of their instruction.

On the other hand, these same tests have not been free from harmful influence. They have overemphasized abstract work and the mechanical elements in arithmetic and algebra at the expense of the applications of the subject; they have also tended to give too much attention to the speed factor, encouraging students to plunge ahead mechanically without thinking through a solution. Perhaps their most detrimental influence has been that they have helped to perpetuate in the course of study certain undesirable types of subject-matter which more progressive teachers have long considered it desirable to eliminate. For example, in a certain algebra test one finds the first example shown below, while a well-known achievement test in arithmetic stresses such atrocities as that shown in the second example. The result has been that many

$$(1)\ \frac{3-2x}{(x-1)^3}+\frac{x+1}{(x-1)^2}-\frac{1}{x-1}=(?)$$

$$(2)\ \frac{17}{18}+\frac{2}{3}+\frac{5}{9}+\frac{1}{6}+\frac{1}{2}$$

teachers, knowing that their pupils are to be examined by some of these tests, have continued to drill upon topics which for some time have been obsolete. Perhaps the chief criticism of most of our tests is that they are too far removed from what is usually being taught in our best modern schools. This is due to the fact that many of these tests were prepared by those who were not specialists in the teaching of mathematics and who had given no study to the modern tendencies in this subject. In constructing a test it is first necessary to determine what subject-matter it is desirable to teach, and then to sc

construct the test that it will measure the pupils' mastery of that particular subject-matter.

Another defect of the testing movement has been that it has not covered all of the instruction in mathematics. In arithmetic and algebra stress has been given to abstract work. While certain tests in problem-solving have also been available, in general they are not satisfactory. The reason for this is that the makers of the problem-solving tests are not quite sure what they want to test. The most neglected field is that of geometry, where thinking and problem-solving predominate and where abstract skills are relatively few. Thus we see that if a modern school should be tested in mathematics with all the tests now available, we would still fall far short of having an adequate measure of all of that school's instruction in mathematics.

One important development growing out of the testing movement has been the creation of practice exercises which aim to remedy the weaknesses disclosed by the standardized tests. Such exercises are now available in both arithmetic and algebra and are serving as an efficient aid to instruction. These practice exercises are far more detailed than any of our standardized tests in their gradation of the difficulties of each topic and hence serve as one of the best means of diagnosing a pupil's difficulties that we now have. In fact, the most promising development in the future seems to be that of making a more effective combination of practice elements and diagnosis in a carefully graded series of exercises, which can easily be corrected and scored by untrained assistants or by the pupils themselves, and which will also provide some flexible standard of accomplishment. Such a type of test will naturally test only the topics that are actually being taught; in fact it is possible for teachers to develop such tests themselves as a measure of the efficiency of their instruction and as an aid in finding the individual weaknesses of the pupils. In such tests the time element should be subordinated to the instructional feature. In fact, as testing programs become

more and more common, speed is being stressed less, while accuracy and understanding are receiving more attention. Even in arithmetic, where the speed element is more important than in algebra or geometry, the rapid development of modern calculating machinery has made a high degree of speed in pencil calculation less necessary than it was formerly.

The future for the testing movement seems bright, and promises to profit by the mistakes of the past. The tests of the next decade will have less of the aspect of the standardized examination and more of the element of being an aid to and a measure of daily classroom instruction.

4. HIGH SCHOOL COURSES IN THE CALCULUS

By VEVIA BLAIR, Horace Mann School for Girls

Elective courses in the calculus are gradually finding their way into the high school. These courses differ in content, arrangement of subject-matter, and method of presentation, in accordance with differing aims of the schools in offering them. Outlines are given below of a few typical courses of this kind.

Wadleigh High School, New York, has for some years been developing a course in the calculus in connection with advanced algebra, trigonometry, and solid geometry. In the second semester of the eleventh year, differentiation and integration of algebraic polynomials and their applications to problems in maxima and minima, and areas are taught in connection with the advanced algebra. Equations of tangents and normals of conic sections are obtained by differentiation, and points of inflection are determined. In the twelfth year, the treatment of trigonometric functions is taken up in connection with the formal study of trigonometry, and algebraic expressions are integrated by trigonometric substitution. The application of integration to volumes and surfaces is made in connection with solid geometry; in particu-

lar, the prismoid is defined as a solid whose cross-section can be expressed as a function of x of degree not higher than three, and the volume is then obtained by integration and applied to the usual elementary solids and to other solids satisfying the definition of the prismoid.

The University High School of the University of California, Oakland, has recently given a combination course in the twelfth year in solid geometry and the calculus. This course follows the recommendation of the National Committee in simplifying the treatment of the mensuration theorems by the use of Cavalieri's and Simpson's theorems and by the elementary methods of the calculus. This course is fully described in the *University High School Journal,* June, 1926.

The Choate School, Wallingford, Connecticut, in 1926–27 introduced a course in the calculus in the twelfth year, using as a text Longley and Wilson, *Introduction to the Calculus* (Ginn & Co.).

The Lincoln School, New York, in 1921 introduced a short unit of the calculus in the eleventh year using as a text Young and Morgan, *Elementary Mathematical Analysis* (The Macmillan Company). This work was discontinued in 1924, when an elective course in the calculus was introduced in the twelfth year. The subject-matter of this course is to be found in Chapters III and IV of Griffin, *Introduction to Mathematical Analysis* (Houghton Mifflin Company), in Chapters VI and VII of Gale and Watkeys, *Elementary Functions and Applications* (Henry Holt & Co.), and in Griffin's *Supplementary Exercises* (Houghton Mifflin Company). The time given to the course is five fifty-minute periods per week for one semester.

Horace Mann School for Girls, New York, offers a course in the tenth year in general mathematics, in which has been included since 1921 a unit of differential calculus. No text is used. The derivative is applied to rates and to problems in maxima and minima. The extent of difficulty is the applica-

tion of the derivative of $f(x) = u^n$ to a problem involving the function $y = \frac{\sqrt{x^2+81}}{3} + \frac{15-x}{4}$. Since 1923 an elective course in the calculus has been offered in the twelfth year. Derivatives are derived for $f(x) = uv$, $f(x) = u/v$, $f(x) = \sin x$ and $f(x) = \cos x$. The extent of difficulty is to be found in the development of the formula $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ Integration is used to find areas, volumes of revolution, liquid pressure, and work done by a variable force. The limit of difficulty is to be found in the calculation of the volume of a ring made by revolving the circle $x^2 + (y-7)^2 = 9$ about the axis of x, the integral of $y = \sqrt{a^2 - x^2}$ being assumed. As the aim of the course is largely cultural, emphasis is placed upon the historical development of the calculus and upon the methods by which its results are obtained. The time given to the course is three forty-minute periods per week for about twenty weeks.

Besides the textbooks mentioned above certain texts published in England may be found available for high school use in this country, such as, for example, *First Steps in the Calculus*, by C. Godfrey and A. W. Siddons (Cambridge University Press), *The Calculus for Beginners*, by J. W. Mercer (Cambridge University Press), *Elements of Differential and Integral Calculus*, by A. E. H. Love (Cambridge U. P.), and *A First Course in the Calculus*, by W. P. Milne, and G. J. B. Westcott (London).

5. COLLEGE ENTRANCE EXAMINATION BOARD

Document 107

Definition of the requirements in Elementary Algebra, Advanced Algebra, and Trigonometry, adopted by the College Entrance Examination Board April 21, 1923:

APPENDIX

Elementary Algebra (Part I)

ALGEBRA TO QUADRATICS

1. The meaning, use, evaluation, and necessary transformations of simple formulas involving ideas with which the pupil is familiar, and the derivation of such formulas from rules expressed in words.[1] *

The following are types of the formulas that may be considered:

$V = \frac{4}{3}\pi r^3$	(the sphere)
$A = \frac{1}{2}h(b + b')$	(the trapezoid)
$s = \frac{1}{2}gt^2$	(falling bodies)
$A = p(1 + rt)$	(amount at simple interest)
$A = p(1 + r)^t$	(amount at compound interest)

2. The graph, and graphical representation in general. The construction and interpretation of graphs.[2]

The following are types of the material adapted to this purpose: statistical data; formulas involving two variables, such as

$$A = \pi r^2,$$

and

$$y = x^2 + 3x - 2;$$

formulas involving three variables, but considered for the case in which an arbitrary value is assigned to one of them, as $V = \pi r^2 h$ for a fixed value, say 4, of h.

3. Negative numbers; their meaning and use.[3]

This requirement includes the fundamental operations with negative numbers and the interpretation of a negative result in a problem, provided such an interpretation is germane to the problem.

4. Linear equations in one unknown quantity, and simultaneous linear equations involving two unknown quantities, with verification of results.[4] Problems.[5]

The coefficients of a single linear equation in one unknown

* The numbers refer to the Notes to Teachers, p. 146.

quantity may be literal fractions. In the case of simultaneous equations, literal coefficients are restricted to simple integral expressions, and to cases readily reducible to such expressions.

5. Ratio, as a case of simple fractions; proportion, as a case of an equation between two ratios; variation.[6] Problems.

6. The essentials of algebraic technique, including:

   a) The four fundamental operations.[7]

   b) Factoring of the following types:[8]

      (1) Monomial factors;

      (2) The difference of two squares;

      (3) Trinomials of the type $x^2 + px + q$.

   c) Fractions, including complex fractions of simple type.[9]

   The requirement includes complex fractions of about the following degree of difficulty:

$$\frac{p+\frac{a}{b}}{q-\frac{c}{d}}, \qquad \frac{\frac{a+3b}{c-5d}}{\frac{a-3b}{c+5d}}, \qquad \frac{\frac{a}{b}+\frac{c}{d}}{\frac{m}{n}-\frac{p}{q}}.$$

   d) Numerical verification of the results secured under *a*), *b*), and *c*).

7. Exponents and radicals.

   a) The proof of the laws for positive integral exponents.

   b) The reduction of radicals, confined to transformations of the following types:

$$\sqrt{a^2b} = a\sqrt{b}, \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b}, \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}},$$

$$\frac{1}{\sqrt[3]{a}} = \frac{\sqrt[3]{a^2}}{a}, \qquad \frac{1}{\sqrt[3]{a^2}} = \frac{\sqrt[3]{a}}{a},$$

   and to the evaluation of simple expressions involving the radical sign.[10]

   c) The meaning and use of fractional exponents, limited

to the treatment of the radicals that occur under *b*) above.

d) A process for finding the square root of a number, but no process for finding the square root of a polynomial.

8. Numerical trigonometry.

The use of the sine, cosine, and tangent in solving right triangles.

The use of four-place tables of natural trigonometric functions is assumed, but the teacher may find it useful to include some preliminary work with three-place tables.

It is important that the pupil should acquire facility in simple interpolation; in general, emphasis should be laid on carrying the computation to the limit of accuracy permitted by the table.

Elementary Algebra (Part II)

QUADRATICS AND BEYOND

1. Numerical and literal quadratic equations in one unknown quantity.[11] Problems.

The requirement includes the solution of the general quadratic equation

$$ax^2 + bx + c = 0,$$

the conditions for the reality and for the distinctness of the roots, and the formulas for the sum and the product of the roots. Simple cases in which x is replaced by z^2 or by a linear binomial, and problems leading to quadratics, are also included; furthermore:

The interpretation of the graph of such an expression as $x^2 - 3x + 5$, meaning thereby the graph of the corresponding equation,

$$y = x^2 - 3x + 5.$$

2. The binomial theorem for positive integral exponents, with applications.[12]

It is not intended under this topic to include problems

involving irrational numbers, or surds, or the expansion of the powers of a binomial having more than one fractional coefficient. Such simple applications as that to compound interest are included.

3. Arithmetic and geometric series.

The requirement is limited to the formulas for the nth term, the sum of the first n terms, the value of such an infinite decreasing geometric series as $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, and to simple applications.[13]

4. Simultaneous linear equations in three unknown quantities.

The coefficients may be integers, numerical fractions, or algebraic monomials.

5. Simultaneous equations, consisting of one quadratic and one linear equation, or of two quadratic equations of the following types:

$$\begin{cases} ax^2 + by^2 = c, \\ xy = h; \end{cases} \qquad \begin{cases} a_1x^2 + b_1y^2 = c_1, \\ a_2x^2 + b_2y^2 = c_2. \end{cases}$$

These may be expressed in other forms, such as

$$x^2 = (r + y)(r - y), \qquad xy = r^2.$$

The coefficients may be integers, numerical fractions, or algebraic monomials.

Graphical treatment is expected in the cases of equations of the types

$$x^2 + y^2 = a^2, \quad x^2 - y^2 = a^2, \quad xy = a, \ y^2 = ax$$

6. Exponents and radicals.

a) The theory and use of fractional, negative, and zero exponents.[14]

b) The rationalization of the denominator in such fractions as

$$\frac{a + \sqrt{b}}{c - \sqrt{d}}$$

c) The solution of such equations as

$$\sqrt{3 - 2x} - x = 30.$$

7. Logarithms.[15]
   a) The fundamental formulas;
   b) Computation by four-place tables;
   c) Applications to the trigonometry of the right triangle.

Advanced Algebra

a) theory of equations

The theorem that an equation of the *n*th degree has *n* roots, if every such equation has one root. Factoring of polynomials in one variable, and the remainder theorem. The coefficients as symmetric functions of the roots. Simple transformations of equations, limited to the removal of the second term, and increase of the roots by a given number and multiplication of the roots by a given factor. Conjugate complex roots of equations with real coefficients.

Equations with whole numbers or fractions as coefficients. Condition for a rational root.[16]

Approximate solution of numerical equations. Descartes's Rule of Signs. Preliminary location of the roots by the graph. Determination of the roots to two or three significant figures.[17]

b) determinants

Definition of determinants of the second and third orders by the explicit polynomial formulas.[18] Evaluation of such determinants by the familiar rules.[19] The simple transformations,[20] proved directly from the definition, and illustrated by examples in which the elements of the determinant are small integers.

Determinants of the fourth order; their evaluation and transformations.[21]

Application to linear equations: (*i*) non-homogeneous

equations in two, three, and four unknowns; (*ii*) homogeneous equations in two and three unknowns; the treatment to include all cases in which the number of equations does not exceed the number of unknowns.[22] The case of compatibility of three non-homogeneous equations in two unknowns is also included.[23]

c) BRIEF TOPICS

Complex numbers, numerical and geometric treatment.[24]
Simultaneous Quadratics.[25]
Scales of Notation.[26]
Mathematical Induction.[27]
Permutations and Combinations. Probability.[28]

TRIGONOMETRY

1. Definition of the six trigonometric functions of angles of any magnitude, as ratios. The computation of five of these ratios from any given one. Functions of 0°, 30°, 45°, 60°, 90°, and of angles differing from these by multiples of 90°.[29]

2. Determination, by means of a diagram, of such functions as $\sin(A + 90°)$ in terms of the trigonometric functions of A.

3. Circular measure of angles; length of an arc in terms of the central angle in radians.

4. Proofs of the following fundamental formulas,[30] and of simple identities derived from them.

a) The Ratio Formulas:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x},$$
$$\cot x = \frac{1}{\tan x};$$

b) The Pythagorean Formulas:

$$\sin^2 x + \cos^2 x = 1, \; 1 + \tan^2 x = \sec^2 x, \; 1 + \cot^2 x = \csc^2 x;$$

c) The Addition Theorems:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y};$$

d) The Double-Angle Formulas, for $\sin 2x$, $\cos 2x$, $\tan 2x$.

5. Solution of simple trigonometric equations of the general order of difficulty of the following:[31]

$$6\sin x + \cos x = 2;\quad \cos 2x = \sin 2x;\quad \tan(x + 30^\circ) = \cot x.$$

6. Theory and use of logarithms, without the introduction of work involving infinite series. Use of trigonometric tables, with interpolation.[32]

7. Derivation of the Law of Sines and the Law of Cosines.

8. Solution of right and oblique triangles (both with and without[33] logarithms) with special reference to the applications. Value will be attached to the systematic arrangement of the work.

Notes to Teachers

It is the purpose of the College Entrance Examination Board to set its requirements in Mathematics in such a way that the work of preparation for them shall be, at the same time, of the greatest intrinsic value to the pupil in connection with his subsequent work in mathematics, physics and engineering, science, economics, and other subjects which make use of mathematics. It appears that this purpose will best be served by specifying the minimum requirement in each subject and leaving to the teacher a free hand for developing in the pupil power, and mastery of the subject. To prepare pupils satisfactorily for the examinations, teachers will, therefore, find it necessary to extend the work somewhat beyond the limits set in each topic.

It is not practicable to specify in advance all the non-mathematical terms that may be used in problems on the

applications of mathematics, and it will never be possible to escape altogether the difficulties arising from the fact that words and expressions which form a part of the language of common life for one group of students may be strange to another group; but it seems worth while to specify that in connection with problems in physics the candidate in Trigonometry will be expected to know the principle of the parallelogram of forces, and for problems involving the points of the compass, although he will not be called upon to interpret such expressions as "northeast by east a quarter east," he will be expected to know the names of the eight principal directions — north, northeast, east, and so on to northwest.

The order in which the topics are listed is not intended to imply any recommendation as to the order of presentation in teaching. For example, the teacher may prefer, in algebra, to begin with work in oral algebra and with problems leading to simple equations.

The formal statement of the requirements is supplemented by the following running commentary.

Elementary Algebra

In conformity with the general principles above laid down, a less extensive treatment of certain purely algebraic topics than has hitherto been the practice in Elementary Algebra is herewith specified, in order that time may be saved for the introduction of intuitive geometry and numerical trigonometry, as well as for better emphasis on the great basal principles of algebra. The examinations are expected to be more searching with respect to the parts of algebra that will be used by the pupil in his later work, and particularly with respect to his ability to solve applied problems, to handle formulas, to interpret graphs, and to solve the types of equations which he will subsequently need to use.

The amount of time and the degree of maturity needed in preparing for these examinations are expected to correspond to the present requirements.

APPENDIX

The requirements in algebra here formulated represent in each case the minimum requirement in that topic. Just as the power of the candidate in geometry is tested by originals not included in any standard list of propositions, so in algebra questions on any topic, which go somewhat beyond the stated requirement, but which are within the power of a candidate properly trained to meet that requirement, may form a part of the examination.

Elementary Algebra (Part I)

ALGEBRA TO QUADRATICS

1. In the work done with formulas, the general idea of the dependence of one variable upon another should be repeatedly emphasized. The illustrations should include formulas from science, mensuration, and the affairs of everyday life. Throughout the course, there should be opportunity for a reasonable amount of numerical work and for the clarification of arithmetical processes.

2. The pupil should be required to construct the graph of such an expression as

$$x^2 - 2x + 3,$$

i.e., to plot the curve

$$y = x^2 - 2x + 3,$$

and to interpret the meaning of any graph constructed.

3. The relation of real numbers to points on a line should be made clear, the cardinal principle being that to every such number corresponds a point on the scale, and conversely. In this way, the relation of positive to negative numbers, of integers to fractional and irrational numbers, and of approximations to the values of both rational and irrational numbers will be rendered intelligible.

4. Beside numerical linear equations in one unknown, involving numerical or algebraic fractions, the pupil will be expected to solve such literal equations as contribute to an

understanding of the elementary theory of algebra. For example, he should be able to solve the equation

$$s = \frac{ar^n - a}{r - 1} \qquad \text{for } a.$$

In the case of simultaneous linear equations, he should be able to solve such a set of equations as

$$\begin{aligned} ax + by &= k, \\ cx + dy &= l, \end{aligned}$$

in order to establish general formulas. But the instruction should include a somewhat wider range of cases, as for example:

$$\begin{cases} ax + (a + b)y = ab, \\ ax + (a - b)y = -ab; \end{cases} \quad \text{or} \quad \begin{cases} ax + by = ab, \\ x + \left(1 + \dfrac{b}{a}\right)y = a. \end{cases}$$

The work in equations will include cases of fractional equations of reasonable difficulty; but, in general, cases will be excluded in which long and unusual denominators appear and in which the highest common factor of the denominators, or the lowest common denominator, cannot be found by inspection.

5. Problems in linear equations, as in ratio, proportion, and variation, will whenever practicable be so framed as to express conditions that the pupil will meet in his later studies. Nevertheless, such studies are generally so technical as to render it impossible to use the phraseology and laws that represent real situations, particularly in connection with proportion and variation. On this account it must be expected that the problems will frequently involve situations that are manifestly fictitious; but even such problems will nevertheless serve the purpose of showing the pupil how to translate from algebraic symbols to ordinary written language, and vice versa, and to use algebraic forms to aid in the solution of problems.

6. It is not considered necessary to treat ratio and proportion as a distinct topic. The pupil is expected to look upon

$$\frac{a}{b}=\frac{c}{d}$$

as a fractional equation, or as a formula involving fractions, and to transform it accordingly. Similarly, he is expected to look upon the equation $y = cx$ as representing simple variation without rendering it obscure by the use of such a form as $y \propto x$. Such terms as "alternation," "composition," etc., should not be introduced. The corresponding theorems, the substance of which can be presented without the use of the technical terms, are of value for some applications in geometry and other branches, but are not included in the requirements in algebra.

7. It is not expected that pupils will be called upon to perform long and elaborate multiplications or divisions of polynomials, but that they will have complete mastery of those types that are essential in the subsequent work with ordinary fractional equations, and with such other topics as are found in elementary algebra. In other words, these operations should be looked upon chiefly as a means to an end.

8. Factoring has been reduced to the cases needed in later work. Under (3), only simple cases, like

$$x^2 - 5x + 6 \qquad \text{and} \qquad x^2 + x - 2$$

are contemplated. A problem may, however, combine two or more of the cases here suggested, as in the polynomial

$$x^2 + 4x - y^2 + 4, \qquad \text{or} \qquad 2ax^2 - 4ax + 2a,$$
$$2(a - b) - 3x(b - a).$$

The teacher may wish to use such examples as

$$a^2 - b^2 + a - b \qquad \text{or} \qquad xy + 2x + 3y + 6$$

for purposes of instruction; but these cases are not included in the requirement.

9. Complex fractions may often be treated advantageously as cases in the division of simple fractions, but usually it is more convenient, in practice, to multiply both numerator and denominator of the complex fraction by a factor that will reduce it to a simple fraction.

The meaning of the operations with fractions should be made clear by numerical illustrations, and the results of algebraic calculations should be frequently checked by numerical substitution as a means to the attainment of accuracy in arithmetical work with fractions.

10. In all work involving radicals, such theorems as $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ may be assumed. Proofs of these theorems should be given only in so far as they make clearer the reasonableness of the theorems; and the reproduction of such proofs is explicitly excepted from the requirements here formulated.

Elementary Algebra (Part II)

QUADRATICS AND BEYOND

11. The coefficients of the equation may be common fractions or simple literal fractions. In such cases it is usually best to begin by freeing the equation of fractions. The method of solution by completing the square should be the one primarily used, with emphasis on the systematic arrangement of the necessary steps and computations. Equations of the form

$$ax^2 + bx = 0$$

should be solved by factoring; beyond this, the method of solution by factoring should be restricted to simple cases, such as

$$x^2 - 5x + 6 = 0, \qquad \text{and} \qquad x^2 + x - 2 = 0.$$

The teacher may wish to use such examples as the factoring of $2x^2 + 5x - 7$ for purposes of instruction; but these cases are not included in the requirement. It should be understood that the recommendations here made do not exclude the method of solution by the formula.

12. The pupil should be able to write the first few terms of the expansion of $(a + b)^n$ for any positive integral value of n, although in general it will be sufficient to take $n \leqq 8$.

13. In connection with the infinite decreasing geometric series, such illustrations as that of the repeating decimal should be used, but the subject of repeating decimals will not be required on the examinations. One important reason for the inclusion of this topic is that it serves as an introduction to the idea of the limit.

At some point in the course, the expression of $a^n - b^n$ as the product of $a - b$ and a second factor should be taken up, and this may be done in connection with the formula for the sum of n terms of a geometric progression. It is sufficient to restrict n to values not exceeding 8. The corresponding factoring of $a^n + b^n$ when n is odd should be included.

14. Fractional, negative, and zero exponents should be presented as cases in which a symbol is defined in such a way as to give universality to a law already established for certain special cases. That is, because

$$a^m a^n = a^{m+n}; \qquad (a^m)^n = a^{mn}; \qquad a^m b^m = (ab)^m,$$

when m and n are positive integers,

$$a^{\frac{p}{q}}, \qquad a^{-m}, \qquad a^0$$

are so defined as to render these laws permanent.

15. The graph of $y = 10^x$ may profitably be used for the purpose of explaining the theory. Only the base 10 need be considered in presenting the subject, but other bases may be used for purpose of illustration.

APPENDIX

Proofs of the theorems

$$\log PQ = \log P + \log Q;$$
$$\log P^n = n \log P;$$

may be required on examinations; but proofs of the derived theorems, such as

$$\log \frac{P}{Q} = \log P - \log Q, \qquad \text{and} \qquad \log \sqrt[n]{P} = \frac{1}{n} \log P,$$

although these can readily be deduced from the first two, will not be required in examinations.

The requirement includes the computation of such an expression as

$$\sqrt[3]{\frac{1.967 + 0.9834}{3.142}};$$

the calculation of s when the formula $v^2 = 2gs$ is given, and values of v and g are given to three or four significant figures; and the complete solution of a right triangle in which the given parts have been measured to four significant figures. It excludes the solution of exponential equations, like $2^x = 3$, and the computation of such expressions as $6.821^{0.1489}$.

Advanced Algebra

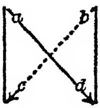
16. The requirement relating to a rational root is restricted to the case in which the coefficients are integers and the coefficient of the highest power of the unknown is unity.

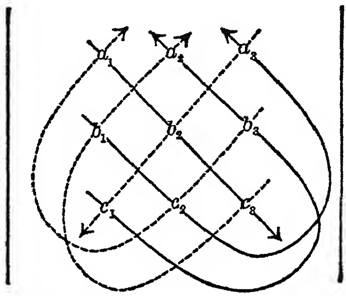
17. After a root has been shown to lie between two values, x_1 and x_2, which are fairly near together, the points of the graph whose x-coördinates are x_1 and x_2 are computed, and the chord determined by these points is drawn. The intersection of this chord with the axis of x yields the next approximation. A graphical solution of the problem may often be used with profit. For closer approximations to the root, this or other methods may be used. But the elaborate developments suggested by the name of "Horner's Method,"

which involve multiplying the roots by powers of 10, constructing a special array for the numerical work (over and above the scheme for reducing the roots by a given number), and finally "contracting" the work are not a part of the requirement.

18. That is, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, and the six-term formula for determinants of the third order.

19. Namely,





20. Such as adding columns, etc.

21. It is important that the pupil should know that a determinant of the fourth order cannot be evaluated by such a scheme as that mentioned under Note 19, but is evaluated by expansion according to the elements of a row or a column. On the other hand, by transformations such as those of Note 20, some elements can be reduced to 0 and the numerical work thus abbreviated.

22. Thus, under (*i*), the pupil will be expected to know in what the solution of the equations

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$

consists, both when the determinant of the coefficients,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

has a value different from 0 and when it equals 0.

In order to avoid too great detail, the requirement is restricted to the case, when the determinant of the equations vanishes, in which at least one (first) minor is different from 0. But the instruction should make clear to the pupil how he can proceed in any numerical case and find the complete solution. Thus, under (*ii*), the solution of the equations

$$\begin{aligned} 3x + 4y - 2z &= 0, \\ 2x + 3y + 4z &= 0 \\ 5x + 7y + 2z &= 0, \end{aligned}$$

consists in giving to z an arbitrary value and then determining x and y in terms of z from the first two equations.

23. It is desirable that the extension of all the foregoing definitions and theorems to more general cases be pointed out; but all such extensions, as also the rule for the multiplication of determinants, are explicitly excluded from the requirement.

24. The reduction of the sum, difference, product, and quotient of two complex numbers to the form $a + bi$. The geometrical construction of the sum and difference, but not of the product and quotient, of two complex numbers.

25. Two equations of the type

$$\begin{aligned} a_1x^2 + b_1xy + c_1y^2 &= d_1, \\ a_2x^2 + b_2xy + c_2y^2 &= d_2. \end{aligned}$$

26. The point of this requirement is the fact that the base 10 is accidental, that any other base greater than 1 could be used, and that computation would be performed by the aid of the addition table and the multiplication table, as is customary with the base 10.

27. A clear exposition of this important method of reasoning. Applications to such problems as summing the series

$$1^2 + 2^2 + 3^2 + \ldots + n^2,$$

and

$$1^3 + 2^3 + 3^3 + \ldots + n^3;$$

the derivation of the formula for

$$(a_1 + a_2 + \ . \ . \ . + a_n)^2;$$

and, possibly, the proof of the binomial theorem for a positive integral exponent; the simpler applications are the more illuminating, and the application to the proof of the binomial theorem will not be required in examinations.

28. Permutations and combinations, restricted to the case of n objects, all of which are different. Probability, restricted to problems of moderate difficulty.

The Brief Topics should really, as the name implies, be treated briefly in the course, and their importance should not be overrated on the question paper by the setting of difficult questions. Their value lies in their mathematical content, rather than in their being a means of further developing technique.

It is not contemplated that the total requirement should be greater than that in Trigonometry or Solid Geometry.

Trigonometry

29. The values of these functions should be read from diagrams, and not made merely a matter of memorizing.

30. This is a minimum list to be memorized. With respect to formulas *c*), the requirement is restricted to the case that x and y are both acute; but the instruction should include the generalization of the formulas to the case of any angles by the addition of 90°.

In restricting explicitly the formulas to be memorized, the object has been to give the pupil perspective and to aid him in an efficient study of the subject. The Board does not wish to discourage the pupil from memorizing certain other formulas when it is convenient for him to do so. But these formulas should hold distinctly a secondary place in the pupil's mind, and they should not be allowed to compete with the fundamental formulas.

31. The instruction should include an explanation of the notation $\sin^{-1}x$ (or arc sin x), $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$ on the basis of the definition:

$$y = \sin^{-1}x, \quad \text{if} \quad x = \sin y,$$

etc., y being measured in terms of radians, and the further definition of the *principal value* of each of these functions should be pointed out. But this topic is not included in the requirement.

32. The use of four-place logarithmic and trigonometric tables is recommended, but both four-place and five-place tables will be furnished the candidates in the examinations of the Board.

The computation of such expressions as $6.821^{0.1489}$, and the solution of exponential equations, like $2^x = 3$, are included in this requirement.

33. This implies the case in which the sides of the triangle are represented by small integers, or in which the data respecting the sides are correct only to two significant figures. This case would naturally be considered at the beginning, before logarithms are introduced, for it contributes to an appreciation of the actual geometrical magnitudes and the relations between them.

Graphical solutions by means of scale and protractor should be freely used as checks.

Both the representation of the angle by degrees, minutes and seconds, and also the decimal subdivision of the degree should be taught, but only one method of dividing the degree is required.

The use of slide rules in the instruction is to be encouraged, but they will not be permitted in the examination for the reason that, when some candidates use slide rules and some do not, it is impossible to mark the papers justly.

APPENDIX

Formulas of Trigonometry

1. The Ratio Formulas:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

2. Formulas read off from a figure; e.g., $\cos(90° + x)$, $\sin(-x)$, etc., in terms of the trigonometric functions of x.

3. The Pythagorean Formulas:

$$\sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x,$$
$$1 + \cot^2 x = \csc^2 x.$$

4. The Addition Theorems:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

5. The Double-Angle Formulas:

$$\sin 2x = 2 \sin x \cos x,$$
$$\cos 2x = \cos^2 x - \sin^2 x,$$
$$= 2 \cos^2 x - 1,$$
$$= 1 - 2 \sin^2 x,$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

6. The Half-Angle Formulas, *a*) in Implicit Form:

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

b) In Explicit Form:

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

7. *a*) Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

b) Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

8. Sums and Products.

$$\sin x + \sin y = 2\sin \tfrac{1}{2}(x+y)\cos \tfrac{1}{2}(x-y),$$
$$\sin x - \sin y = 2\sin \tfrac{1}{2}(x-y)\cos \tfrac{1}{2}(x+y),$$
$$\cos x + \cos y = 2\cos \tfrac{1}{2}(x+y)\cos \tfrac{1}{2}(x-y),$$
$$\cos x - \cos y = -2\sin \tfrac{1}{2}(x+y)\sin \tfrac{1}{2}(x-y).$$

9. Special Formulas for Triangles:

a) Law of Tangents:

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

b) For the case in which the three sides are given:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \tfrac{1}{2}(a+b+c).$$

Formulas 1, 3, 4, and 5 (and also 7) should be memorized, like the multiplication table. The proofs of 1 and 3 are immediate. In the case of the addition theorems, the proof for

the sine and the cosine of the sum of two acute angles should be given and the formulas extended to the case of any angles by adding $90°$ to x or y. As already noted, the proof will be required on examinations only for the case in which x and y are both acute. The third formula, 4, and the formulas, 5, follow at once from these. The formulas for $\sin (x - y)$, etc., are obtained by replacing y by $-y$ and making the obvious reductions. These formulas need not be memorized, but should be deduced when required.

Formulas 2 include $\sin (x \pm 90°)$, $\sin (90° - x)$, $\sin (x + 180°)$, $\sin (180° - x)$, $\sin (-x)$, etc., and the corresponding cases for the cosine, tangent, and cotangent. None of the formulas should be memorized, but each should be read off from the appropriate figure, which the pupil soon comes to visualize without pencil and paper.

The half-angle formulas, 6, should be deduced when needed from the double-angle formulas, 5. The double signs in 6 *b*) depend on the magnitude of the angle and cannot be grouped automatically.

Formulas 7 should be memorized, as already noted, and the pupil should be familiar with the proofs.

Formulas 8 are not to be memorized, although they are occasionally used in practice, and it is well for the pupil to know that such formulas exist. Their proof affords a useful exercise in the manipulation of the trigonometric identities.

Any triangle can be solved by the law of sines and the law of cosines, and these formulas are to be preferred when the sides are represented by simple numbers. Formulas 9 are well adapted to logarithmic computation in the cases to which they apply. The pupil should be able to use them, or formulas equivalent to them; but he is not required to memorize them, or to learn their proof. The formula for the area of a triangle when two sides and the included angle are given, namely,

$$\text{Area} = \tfrac{1}{2}\, ac \sin B,$$

is not included, since it should be read off, whenever needed, immediately from the figure.

APPENDIX

6. COLLEGE ENTRANCE EXAMINATION BOARD

Document 108 (Extract)

Definition of the requirements in Plane Geometry, Solid Geometry, Plane and Solid Geometry, Major Requirement, and Plane and Solid Geometry, Minor Requirement:

SYLLABI IN GEOMETRY

The following syllabi have been prepared to indicate the scope of the requirements in geometry. It will be seen that the content of the standard requirements in plane geometry and in solid geometry is essentially the same as heretofore. A redistribution of emphasis on the various parts of the work, designed to lighten the demands on the candidate's memory and to give increased opportunity for attention to the development of geometrical understanding, is explained in subsequent paragraphs of this introductory statement.

To meet the situation created by the gradual disappearance of solid geometry from the schools, the present requirements (Mathematics C, D, and CD) have been so modified as to permit teachers greater freedom in the treatment of the material, and a new requirement (Mathematics cd), covering selected parts of plane and solid geometry and calling for about as much preparation as is represented by the present requirement in plane geometry, has been added.

As in the past, the examinations will consist partly of questions on *book propositions* and partly of *originals*. In the former type of question, the candidate will be asked to give demonstrations of standard theorems which are assumed to have been presented to him in his course of study, or to reproduce standard constructions. Under the latter type are included the demonstration of theorems which are not assumed to be familiar to the candidate, problems of measurement and calculation, and problems in the working out of unfamiliar constructions and the identification of unfamiliar loci. Questions calling for simple geometrical knowledge and understanding may fall under either type.

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An important novelty in the present statement of requirements relates to the treatment of book propositions. These will be chosen, not from the entire syllabus at large, but from a restricted list of propositions, designated in each syllabus by asterisks. The starred propositions have been chosen partly with regard to their intrinsic importance, but partly also with regard to their availability for examination purposes. A theorem may be of the highest importance, and yet may have a proof which, for one reason or another, is of such a nature that little information as to the candidate's ability is obtained by asking him to reproduce it. On this account, some of the most important propositions in the course, such as those on the circumference and area of the circle, are omitted from the starred list.

In the case of the unstarred propositions, the candidate will be expected to be familiar with their content, so as to be able to answer questions about their substance or use them as a basis for solving originals, but he will not be assumed to have their demonstrations in mind. A few propositions have been included on account of their general importance and interest without reference to direct or indirect use for examination purposes. The demonstration of an unstarred proposition may sometimes be called for on an examination, but only if it is of such a nature that the candidate can reasonably be expected to work it out as an original. It will undoubtedly be desirable to present the proofs of most of the unstarred propositions in regular sequence in the course, formally or informally, as one of the best supplements to the instruction directly needed for the satisfaction of the minimum requirements. It is intended, however, that it shall be most advantageous for the pupil, in preparing for the examinations, to spend his time in developing facility in independent thinking on geometrical topics, rather than in memorizing such proofs.

About one third of the examination will be devoted to book propositions from the starred list, except in the case of solid geometry as explained later. The remainder will be divided

between easy originals which ought to be within the reach of any properly trained candidate, and originals which, though still not of excessive difficulty, are designed to test the powers of the better candidates. The easy as well as the more difficult originals may be unfamiliar theorems, or problems in mensuration, construction, or loci. Whether papers so planned are found to be intrinsically easier or harder than those that have been set in the past, it is intended that the marking shall be so administered that the difficulty of obtaining a passing grade is not materially changed.

The questions constituting the examination (that is, the main subdivisions of the paper, numbered from 1 to 6, more or less) will as a rule be weighed equally by the readers; but a bonus may be given for exceptional work on a given question, and, on the other hand, the general character of a candidate's paper may indicate serious deficiencies, even though a considerable total be scraped together from his fragmentary answers to many questions. The readers will appraise a book as a whole, and not cling to any mechanical method of marking when occasion demands the exercise of human judgment. The questions may be answered in any order, and it is advisable that the candidate make sure of those which he can answer best. When a single question is composed of two or more parts, the relative weights to be assigned to the parts will be determined by the readers according to circumstances.

Each of the following syllabi has been arranged in what is regarded as a practicable teaching order, but this order is in no wise prescribed. The candidate may base his answers on any sequence of propositions which is logically sound, and may even quote propositions not listed in the syllabus at all, if they have been presented as standard propositions in his course. The readers occasionally have to use their discretion in deciding whether an answer made unduly easy by the citation of an unusual reason deserves full credit. For example, if a candidate proves the starred theorem *23 in Plane Geometry merely by deducing it from theorem 24 or 25, he may

reasonably be expected to make up for a considerable saving of time here by a higher standard of performance on the rest of the paper than would otherwise be required.

For convenience of reference, formulas have been appended to some of the theorems in mensuration. The notation used in these formulas is not prescribed, but if the candidate uses a different notation, he should be sure that its meaning is unmistakable.

It is to be emphasized once more that the syllabi have been prepared as a basis for the conduct of examinations, not as a recommendation for the content and arrangement of ideal courses. Many of their features, especially in connection with the distribution of asterisks, have been determined by the peculiar exigencies of examination technique.

In particular, it will be noticed that proofs involving the theory of limits are not required for examination purposes. This is for the reason that the definition and maintenance of a satisfactory standard of performance is difficult for examiners and readers, as well as for teachers and candidates. Apart from the examination requirement, it is in the interest of sound mathematical training that each class of pupils receive the best and most thorough instruction in the theory of limits that they can accept. An immeasurably improved understanding of the meaning of limits has been perhaps the most important achievement of mathematical science in the past hundred years. As this improved knowledge becomes generally available, the importance of the subject for elementary instruction will increase rather than diminish. In the light of developments already in progress, there is reason for the belief that the present period of transition, in which the setting of any kind of uniform standards appears impracticable, will not be of long duration.

Plane Geometry

The significance of the starred and the unstarred propositions has been explained above.

APPENDIX

The originals on the examination will in general depend for their solution on propositions mentioned in the syllabus. But occasionally an original will be so framed that a solution will occur more readily to the candidate who is familiar with such important geometrical facts as the properties of the 30° and the 45° right triangles.

With regard to constructions, the candidate is expected to be able to perform and to describe accurately those listed at the end of the syllabus, and also, as originals, others based on these. He is not required to give proofs of constructions, unless a proof is specifically called for by the question, and such proofs will not be regarded as constituting a part of the book-work requirement, but will have the status of originals. The candidate is expected to be provided with ruler and compasses, and to use them. In default of these instruments, however, he will receive credit for a satisfactory free-hand sketch showing all construction lines.

Solid Geometry

The work in solid geometry may well be supplemented by a considerable amount of material which remains outside the scope of the examination requirement. The importance of limit proofs has already been emphasized. Other valuable material is contained in the theorems of Cavalieri and Euler, and the so-called prismoid or prismatoid formula.

A few theorems on similar solids have been included in the syllabus, unstarred. The candidate should be familiar with their substance, as a basis for problems in measurement and calculation, but they should not be made the occasion of a large amount of demonstrational work.

The candidate should know what is meant by the perpendicularity of skew lines, but will not be required to deal with the measure of an angle between two skew lines which are not perpendicular to each other.

Attention is again called to the fact that the order in which the propositions are listed is in no wise to be regarded as a

prescription of the order in which the proofs shall be taken up. Thus, for example, the sphere may be treated early in the course, and the treatment of dihedral angles based on the geometry thus developed.

The Board wishes to accord all due latitude in the treatment of the subject of solid geometry. It recognizes the value of the further training in logical demonstration which supplements the study of plane geometry and is given in standard courses at the present time. It recognizes also that the intuitive geometry of the early school course may well be carried further as regards both a firmer grasp on space relations and the visualization of space figures, and the mensuration of surfaces and solids in space.

The examinations will be constructed with reference to this larger interpretation of the requirement. In the past, the candidate has been expected to answer six questions, and this will be assumed for convenience in defining the nature of the new examinations. These papers will consist of seven questions, of which the candidate will be expected to answer six. Two of these questions will call for demonstrations of propositions from the starred list, but not both of these propositions will be chosen from Book VI. Many teachers have felt that the amount of formal demonstration demanded by this Book has been excessive and has obscured the subject-matter. The purpose of the new requirement is to give the teacher freer hand, enabling him, if he so desires, to teach the facts concerning the relations of lines and planes in space by means of problems and constructions.

In connection with the drawing and measurement of geometrical solids, the candidate should be thoroughly familiar with the terminology of the subject, including such terms as the following: polyhedron; regular polyhedron; diagonal; face diagonal; prism, right prism; oblique prism; truncated prism; parallelepiped; right parallelepiped; rectangular parallelepiped; pyramid; frustum of a pyramid; vertex; lateral edge; slant height; altitude; cylinder and cone, and the customary

terms relating to them, defined with complete generality, though restricted in problems of mensuration to the right circular cylinder and cone; zone; lune, great circle, small circle, segment of one base and sector of a sphere.

Problems of mensuration, stated in words or in terms of a figure, form an important topic of the course. The candidate is expected to be familiar with the fundamental formulas of mensuration in plane geometry, with the use of natural sines, cosines, and tangents in solving right triangles, as taught in numerical trigonometry, and with the formulas of solid geometry enumerated in the Appendix to the Syllabus.

Another topic explicitly included is the solution of locus problems, to the extent of visualizing, describing, and representing the figure on paper; formal proofs are not in general required.

With the considerable reduction in the formal proofs demanded by the present requirement, it is expected that candidates will show greater facility in visualizing space figures and in representing such figures on paper. Moreover, a firmer grasp of the facts of solid geometry is contemplated, and problems in mensuration which test all of these things will form a part of the examinations. The type of training expected of a candidate, so far as this new material is concerned, is indicated by a list of problems in the Appendix to the Syllabus.

Plane and Solid Geometry

MAJOR REQUIREMENT

This requirement is based on the syllabi for the two preceding requirements, taken together, and corresponds to the present requirement, Mathematics *CD*, 1½ units.

Plane and Solid Geometry

MINOR REQUIREMENT

This requirement is designed to cover the most important parts of plane and solid geometry, in such a way that the preparation for it can be completed in the time usually de-

voted to the standard requirement in plane geometry. It is designated as Mathematics *cd*, and counts 1 unit.

The notes bearing on the syllabi for the preceding requirements, with regard to originals and constructions and the matter of skew lines, are applicable here.

A number of theorems have been omitted from the present syllabus which would be used in proving some of the unstarred theorems that are included, so that the latter theorems will necessarily be treated somewhat informally, if the course is not to be made unduly long. The preliminary propositions that are needed for proving the starred theorems have been retained.

An omitted theorem may be used as an original on an examination, if it is of suitable difficulty. And an original may involve a simple omitted theorem incidentally — for example, the corollary that, if two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary — but then the fact that the candidate has to recognize and prove the point in question will be taken into account in estimating the difficulty of the problem.

[Here follow lists of theorems and constructions in Plane Geometry (89 theorems, 20 constructions), 92 theorems in Solid Geometry, theorems and constructions in Plane and Solid Geometry, Minor Requirements (96 theorems, 19 constructions). The theorems and constructions of these lists are indicated in the National Committee's lists, pp. 79–91, by placing after each theorem or construction the number in square brackets [ ] of the corresponding theorem or construction of the C.E.E.B. list. The starred propositions are also indicated by placing an asterisk after the corresponding number. The designation *cd* placed after a theorem or construction on pp. 79–91 indicates that it is included in the C.E.E.B. requirement for Plane and Solid Geometry, Minor Requirement (Mathematics *cd*). All propositions of the C.E.E.B. lists not contained in the National Committee's lists are given below, with their respective numbers.]

APPENDIX

Plane Geometry

Two right triangles are congruent if the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other. [9, cd]

Through any three given points not lying in a straight line one circle, and only one, can be drawn. [36*]

If tangents to a circle from an external point are drawn, they make equal angles with the line joining the given point to the center. [45, cd, part]

An angle formed by a secant and a tangent, intersecting at a point outside the circle, is measured by half the difference between the intercepted arcs. [52, part]

A circle can be circumscribed about any regular polygon. [78*]

A circle can be inscribed in any regular polygon. [79]

Regular polygons of the same number of sides are similar. [82]

If the number of sides of a regular polygon inscribed in a circle be increased indefinitely, the apothem of the polygon will approach the radius of the circle as its limit. [84]

The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely; and the area of the circle is the limit of the areas of these polygons. [85]

The areas of two circles are to each other as the squares of their radii. [89]

Solid Geometry

Through a given point in a given line one plane, and only one, can be passed perpendicular to the line. [4, cd]

Through a given external point one plane, and only one, can be passed perpendicular to a given line. [5, cd]

If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane. [11, cd]

If two lines are parallel to a third line, they are parallel to each other. [12, cd]

Through either of two skew lines one plane, and only one, can be passed parallel to the other line. [14, cd]

Through a given point in space one plane, and only one, can be passed parallel to each of two skew lines, or else parallel to one line and containing the other. [15, cd]

APPENDIX

A line perpendicular to one of two parallel planes is perpendicular to the other also. [18, cd]

Through a given point outside a plane one plane, and only one, can be passed parallel to the given plane. [19, cd]

If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane. [20, cd]

If two planes are perpendicular to each other, a line perpendicular to one of them at any point of their intersection will lie in the other. [24, cd]

If two planes are perpendicular to each other, a line drawn perpendicular to one of them through any point of the other will lie in the second plane. [25, cd]

Through a given line not perpendicular to a given plane, one plane and only one can be passed perpendicular to the given plane. [28, cd]

The locus of points equidistant from the vertices of a triangle is the line through the center of the circumscribed circle, perpendicular to the plane of the triangle. [31]

Two prisms are congruent if three faces which include a trihedral angle of one are respectively congruent to three faces which include a trihedral angle of the other, and are similarly placed. [38]

Two right prisms having congruent bases and equal altitudes are congruent. [39]

The areas of two similar polyhedrons are to each other as the squares of any two corresponding edges. [60]

The areas of two similar cylinders, or of two similar cones, are to each other as the squares of any two corresponding lines. [62]

The volumes of two similar cylinders, or of two similar cones, are to each other as the cubes of any two corresponding lines. [63]

Through any two given points on the surface of a sphere an arc of a great circle can be drawn. [65]

Through any three given points on the surface of a sphere one circle and only one can be drawn. [66]

An angle formed by arcs of two great circles is measured by the arc of a great circle described from its vertex as a pole and included between its sides, produced if necessary. [73*]

If two sides of a spherical triangle are equal, the angles opposite these sides are equal. [84]

If two angles of a spherical triangle are equal, the sides opposite these angles are equal. [85]

APPENDIX

If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater. [86]

If two sides of a spherical triangle are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater. [87]

A spherical triangle is equivalent to a lune whose angle is half the spherical excess of the triangle. [88]

A convex spherical polygon is equivalent to a lune whose angle is half the spherical excess of the polygon. [89]

The shortest line that can be drawn on the surface of a sphere, connecting two given points of the sphere, is an arc of a great circle. [90]

[Document 108 also contains an Appendix in which will be found valuable suggestions to teachers on "Formulas of Mensuration for Solids," "Drawing," and a list of "Problems."]

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